# A primer to geometric group theory

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This text, originally written as an introduction to my book on random groups, gives an introduction to the study of discrete groups from a geometric point of view. It contains basic facts on group presentations and free groups; the construction of Cayley graphs and complexes; a sketch of what van Kampen diagrams are; and the different definitions of hyperbolic groups. An explicit example then gathers it all in two pictures.

We do not pretend to give an overview of geometric group theory as a whole, and many important aspects of the field are not mentioned.

A few selected references are given at the end as suggested reading.

#### Free groups, group presentations

From a geometric group theorist's viewpoint, which may not be everyone's, the simplest of all groups are free groups over some set of generators.

Let  $S_+$  be any set (frequently  $S_+ = \{a_1, \ldots, a_m\}$  is finite). Intuitively speaking, the free group on  $S_+$  is the group consisting of all formal products of elements of  $S_+$  and their formal inverses, with the cancellation  $xx^{-1} = x^{-1}x = e$  as the only computation rule.

More precisely, let  $S_{-}$  be the set of formal inverses  $x^{-1}$  of the elements  $x \in S$ , which is just a distinct copy of  $S_{+}$ , and let  $S = S_{+} \sqcup S_{-}$ . For  $n \in \mathbb{N}$ , a word of length n on the alphabet S is a sequence of n elements of S. In the particular case n = 0 there is only one such sequence, called the *empty word*. A word on S is just a word of any length, i.e. an element of  $\bigsqcup_{n=0,1,2\dots} S^n$ .

A word is said to be *reduced* if it does not contain as a subword any sequence of the form  $xx^{-1}$  or  $x^{-1}x$ , for any  $x \in S$ . With any word can be associated a reduced word, by iterated removal of all such cancellable pairs (the reduced word obtained

does not depend on the order in which removals are performed), an operation called *reduction*.

The free group generated by S (or by  $S_+$ , terminology is floppy) has all reduced words on S as its elements. Multiplication is simply the concatenation of words, followed by reduction if necessary. The neutral element e is the empty word.

Of special importance is the case when  $S = \{a_1, \ldots, a_m\}$  is finite; the corresponding free group is denoted by  $F_m$ . When m = 1 it is isomorphic to the group of integers  $\mathbb{Z}$ .

Now any group can be seen as a free group but with more "computation rules" than simply  $xx^{-1} = x^{-1}x = e$ . This gives rise to the notion of group presentations: a group specified by a given set of generators S, with some "enforced" computation rules. For example, the presentation  $\langle a \mid a^3 = e \rangle$  (read: the group generated by a, knowing that  $a^3 = e$ ) defines a group isomorphic to  $\mathbb{Z}/3\mathbb{Z}$ .

Namely, a computation rule is any equality  $w_1 = w_2$  where  $w_1, w_2$  are words on a given alphabet S. Any such rule can be rewritten  $w_1w_2^{-1} = e$ , and so most of the time rules are specified by giving only one word r, with the rule r = e in mind.

So let R be a set of words on a given alphabet  $S = S_+ \sqcup S_-$ . The group presented by  $\langle S_+ | \{r = e\}_{r \in R} \rangle$ , or, more simply, by  $\langle S_+ | R \rangle$ , is defined as follows. Start from the free group  $F_S$  generated by  $S_+$ . The way to enforce a relation r = e is to quotient  $F_S$  by the normal subgroup generated by r. So let  $\langle R \rangle$  be the normal closure of the subgroup of  $F_S$  generated by all words  $r \in R$ . The group presented by  $\langle S_+ | R \rangle$  is the group  $F_S / \langle R \rangle$ . It is the "largest" group in which all relations  $r = e, r \in R$ , hold.

Let us give a few examples: the free group  $F_1$  of rank one (a.k.a.  $\mathbb{Z}$ ) has the presentation  $\langle a \mid \emptyset \rangle$ . The cyclic groups  $\mathbb{Z}/n\mathbb{Z}$  are given by  $\langle a \mid a^n = e \rangle$ . The free group of rank two is  $F_2 = \langle a, b \mid \emptyset \rangle$ ; if we force a and b to commute we get  $\mathbb{Z} \times \mathbb{Z} = \langle a, b \mid ab = ba \rangle$ .

The elements of  $S_+$  in a presentation  $\langle S_+ | R \rangle$  are called *generators*. Those of R are called *relators*. It is easily seen that relators can always be assumed to be reduced words.

Note that any group has some presentation, in a kind of tautological way. Let G be a group and take  $S_+ = G$  i.e. all elements of G will be generators. Now let the set of words R consist of all products of elements of S which happen to be equal to e in G. It is easy to check that  $G = \langle S_+ | R \rangle$ . (Of course this presentation is, in general, way too large.)

This means that free groups have a "universal" property, namely, for each group G there is a set S and a surjective homomorphism from a free group  $F_S$  to G. More precisely, if  $X \subset G$  is any set which generates G together with  $X^{-1}$ , then there is a surjective homomorphism  $F_X \to G$  "sending X to X" i.e. sending an abstract word

in the generators to its image in G. If  $\mathcal{R}$  is the kernel of this homomorphism, then any part  $R \subset \mathcal{R}$  the normal closure of which is  $\mathcal{R}$ , will give rise to a presentation  $G = \langle X | R \rangle$ .

A group is finitely presented if it admits a presentation  $\langle S_+ | R \rangle$  with both  $S_+$ and R finite. A group is finitely generated (or of finite type) if  $S_+$  is finite.

Presentations of a given group are by no means unique. For example, the trivial group  $\{e\}$  has arbitrarily (not so) stupid presentations such as  $\langle a, b \mid a = e, b^2 a b^{-1} = e \rangle$ . (In fact it is not even algorithmically decidable whether a given presentation defines the trivial group or not!)

#### Cayley graphs and complexes

Hereafter, in order to get locally compact objects, we suppose that all sets of generators S are finite.

**Cayley graphs.** The above may seem quite combinatorial but actually carries a geometrical meaning, which is sometimes a more natural way to think of group presentations. For example, the group given by  $\langle a \mid a^3 = e \rangle$  can be thought of as a three-edge cycle.

Let G be a group generated by a set  $S = S_+ \sqcup S_-$ . The Cayley graph of G with respect to S is the graph with elements of G as vertices and in which edges correspond to multiplication on the right by the generators.

More precisely, the Cayley graph is an unoriented graph with some decoration. Vertices are the elements of G. Now for each  $x \in G$  and  $s \in S_+$ , add an unoriented edge between the vertices x and xs, so that edges are in bijection with  $G \times S_+$  (this may result in multiple edges and loops). Now keep track of this on the edges, by deciding that with each edge together with an orientation choice, will be associated a label in  $S_+ \sqcup S_-$ ; namely, the edge from x to xs, oriented this way, will have label s, and label  $s^{-1}$  when oriented the other way around<sup>1</sup>.

Basic examples are (w.r.t. the obvious generating sets): The Cayley graph of the free group  $F_m$  is an infinite tree in which each vertex has valency 2m. The Cayley graph of the group  $\mathbb{Z} \times \mathbb{Z}$  is an infinite square grid in the plane. The Cayley graph of the cyclic group of order n is an n-edge cycle.

The Cayley graph is homogeneous: all vertices play the same role and could have been chosen to represent the neutral element e.

Given any vertex x in the Cayley graph and any word w on the alphabet S, we can start at x and track w in the graph by "following the edge labels", which

<sup>&</sup>lt;sup>1</sup>The details of these definitions guarantee that each vertex has valency  $2 \times \#S_+$  and that the Cayley graph of the cyclic group of order *n* is an *n*-cycle even for n = 1, 2.

brings us at xw. This path is called the *lift* of w (starting at x). Note that w is reduced if and only if this path has no local backtracks.

Left and right actions. The group G acts on the vertices of the Cayley graph in two ways: by left or right multiplication.

Left multiplication by  $g \in G$  is a graph action: it brings adjacent vertices to adjacent vertices, preserving edges. In particular it preserves the graph distance. Note however that it does not bring a point to a nearby point: x and gx can be very far away in the graph.

Conversely, right multiplication by  $g \in G$  is not, in general, a graph action, because a priori one does not pass from xg to xsg by multiplication by s. So two points linked by an edge are not mapped to two points linked by an edge: right multiplication acts only at the level of vertices of the Cayley graph. However, right multiplication by g moves a given point by a distance at most the length of g (since it corresponds to following the path labelled by g in the Cayley graph, starting at the given point).

Consequently, when one mentions the action of G on its Cayley graph, it is the left action which is meant. (Test all this in  $F_2...$ )

**Cayley complexes.** It is worth noting that any word equal to e in G will lift to a loop in the Cayley graph, and conversely. If G is given by a presentation  $\langle S_+ | R \rangle$ , this will be the case, in particular, for the relators  $r \in R$ .

The Cayley complex of a presentation  $G = \langle S_+ | R \rangle$  is a 2-dimensional complex obtained by gluing a disk on all paths of the Cayley graph labelled by a relator  $r \in R$ . More precisely, to each  $x \in G$  and  $r \in R$ , consider the lift of r starting at x in the Cayley graph, which is a closed path, and glue a disk along this path. Consequently the set of faces of the Cayley complex is in bijection with  $G \times R$ .

Basic examples are (w.r.t. the usual presentations): The Cayley complex of a free group is just its Cayley graph. The Cayley complex of  $\mathbb{Z} \times \mathbb{Z}$  is the square tiling of the plane. The Cayley complex of the cyclic group of order n consists of n disks sharing a common boundary (a copy of the relation  $a^n = e$  is glued to each element)<sup>2</sup>.

Most importantly, the Cayley complex is simply connected. Indeed, loops in the Cayley graph label words equal to the identity. Since by definition, relators generate all relations in the group, such words are exactly products of (conjugates

<sup>&</sup>lt;sup>2</sup>Several authors, including Lyndon–Schupp (1997), take another convention when some relator is a proper power, in order to avoid this multiplicity. This sometimes leads to contradictions in their exposition, such as in Prop. III.4.3 of Lyndon–Schupp. With the convention used here, the Cayley complex is the universal covering of the 2-skeleton of the standard classifying space associated with the presentation.

of) relators in the presentation. We precisely added disks along loops of the Cayley graph representing the relators, making these loops homotopically trivial.

A word about classifying spaces. Here is another method to define the Cayley graph and complex. Namely, consider a group presentation  $G = \langle S_+ | R \rangle$ . There is a standard way to get a topological space with fundamental group G. Start with a bouquet B of  $\#S_+$  circles, that is, a graph made of one single vertex (denoted e) and  $\#S_+$  unoriented loops. Choose an orientation for each loop and label loops bijectively by the generators in  $S_+$ ; consider that the loop with reverse orientation bears the inverse label in  $S_-$ .

The fundamental group of this bouquet of circles is the free group with  $\#S_+$  generators. The universal cover of this labelled graph is exactly the Cayley graph of the free group on  $S_+$ .

Now add "relations" in the following way. Any word on the alphabet  $S_+ \sqcup S_$ lifts to a closed path in the labelled graph B, just as above for Cayley graphs. For each relator  $r \in R$ , add a disk to B along the path labelled by r. In the fundamental group of B, this has the effect of killing the element r. Thus at the end, one gets a 2-complex B (with one vertex,  $\#S_+$  edges and #R faces) the fundamental group of which is precisely the group  $G = \langle S_+ | R \rangle$ . The universal cover of B is exactly the Cayley complex of this presentation.

(By the way, if one goes on with this process and kill all the higher-dimensional homotopy groups of B by adding sufficiently many balls of dimension  $\geq 3$ , one gets a so-called *classifying space* for the group G, i.e. a space BG with fundamental group G and such that the universal cover EG is contractible.)

### Groups as geometric objects

Here again all groups are assumed to be finitely generated.

Metrics on groups. The Cayley graph of a group w.r.t. some generating set is naturally a metric space, defining each edge to have length 1.

The combinatorial way to look at this is as follows: Given a group G generated by a set of elements  $S_+$ , it is natural to define the norm ||g|| of an element  $g \in G$  as the smallest length of a word expressing g as a product of generators in  $S_+ \sqcup S_-$ . This coincides, of course, with the graph distance from g to e in the Cayley graph of G w.r.t. this generating set. Note that  $||g|| = ||g^{-1}||$ , and of course  $||gg'|| \leq ||g|| + ||g'||$ .

So we get a distance function on G by setting dist(g, h) to be the graph distance from g to h in the Cayley graph. Since edges of the Cayley graph correspond to right multiplication by a generator, this is the smallest length of a word w such that h = gw, that is,  $dist(g, h) = ||g^{-1}h||$ . The two properties of  $||\cdot||$  mentioned above are just the usual distance axioms.

As mentioned above, the left action of G on itself preserves this metric.

**Changing generators.** Of course this distance depends on the chosen generating set. A finitely generated group is thus not equipped with a canonical metric, but with a family of metrics associated with all possible finite generating sets. These metrics are related in some way, which we explore now.

Let  $S_+ = \{s_1, \ldots, s_n\}$  and  $S'_+ = \{s'_1, \ldots, s'_p\}$  be two finite generating sets for a group G; let  $\|\cdot\|$  and  $\|\cdot\|'$  be the two associated norms. We want to control  $\|\cdot\|'$  in terms of  $\|\cdot\|$ .

Since  $S_+$  is a generating set, each of the elements in  $S'_+$  has a writing in terms of the generators in  $S_+ \sqcup S_-$ . Let K be the largest length of such a writing i.e.  $K = \max_{s' \in S'_+} ||s'||$ . Since  $S'_+$  is finite, we have  $K < \infty$ .

Now suppose that some element g has a writing of length n in terms of  $S'_+ \sqcup S'_-$ . By replacing each element of  $S'_+$  by a writing of it in terms of  $S_+ \sqcup S_-$ , we get a writing of g of length at most Kn in terms of  $S_+ \sqcup S_-$ . So  $\|\cdot\| \leq K \|\cdot\|'$ . The reasoning is two-sided, and so we get that the metrics defined by  $S_+$  and  $S'_+$  are bi-Lipschitz equivalent.

**Quasi-isometries.** Actually a slightly looser definition of equivalence between metric spaces than bi-Lipschitz equivalence has proven very fruitful: that of *quasi-isometry*. It allows, for example, the spaces  $\mathbb{R}$  and  $\mathbb{Z}$  to be quasi-isometric, by neglecting what happens at small scales.

Let  $(X, d_X)$  and  $(Y, d_Y)$  be two metric spaces. They are *quasi-isometric* if there exist two maps  $f : X \to Y$  and  $g : Y \to X$  which distort distances in a linearly controlled way and which are almost inverse to each other (up to a finite distance error). That is, there exist constants  $\lambda > 0, C \ge 0$  such that

$$d_Y(f(x), f(x')) \leq \lambda \, d_X(x, x') + C$$
  

$$d_X(g(y), g(y')) \leq \lambda \, d_Y(y, y') + C$$
  

$$d_X(g(f(x)), x) \leq C$$
  

$$d_Y(f(g(y)), y) \leq C$$

for all  $x, x' \in X, y, y' \in Y$ . This is an equivalence relation.

This notion is relevant for unbounded spaces only: any bounded metric space is quasi-isometric to a point.

A change in the generating set of a group is in particular a quasi-isometry. So any quasi-isometry invariant of a metric space, when applied to Cayley graphs, will provide a well-defined invariant of finitely generated groups.

### Van Kampen diagrams

Van Kampen diagrams are a visual way to represent how all equalities holding in a group are derived from combinations of relators.

Let  $G = \langle S_+ | R \rangle$  be a group presentation. Since we only have access to elements of G as products of generators, we want to know when two words represent the same element of G, i.e. we are interested in the set of equalities of words x = ythat hold in G. Since this can be rewritten as  $xy^{-1} = e$ , it is enough to determine the set of words representing the identity element of G (so-called *word problem*).

This problem is actually algorithmically unsolvable: there is no algorithm that, given any finite presentation and any word, always answers whether the given word is equal to e in the presentation. Moreover, this already holds for some fixed groups: there exist some group presentations for which there is no algorithm which, given any word, answers whether or not it is equal to e in the presentation.

A word is equal to e in the group G presented by  $\langle S_+ | R \rangle = F_{S_+}/\langle R \rangle$ , by definition, if it lies in the kernel of the map  $F_{S_+} \to G$ , that is, in the normal subgroup  $\langle R \rangle$  generated by the relators. Hence, a word w is equal to e in G if and only if, as a word, it can be written as a product of conjugates of relators:

$$w = \prod_{i=1}^{N} u_i \; r_i^{\pm 1} \, u_i^{-1}$$

where  $r_i \in R$  and  $u_i$  is any word. (The number N of relators in this decomposition, which depends on w, will play an important role below in the definition of hyperbolic groups.) So algebraically speaking, the set of consequences of the relators is the normal closure of the subgroup they generate. Van Kampen diagrams are a visual interpretation of these products of conjugates.

Giving a topologically clean definition of van Kampen diagrams is beyond the scope of this review. Basically, the idea is to consider each relator  $r \in R$  as a polygon with as many edges as letters in r, the edges of which are labelled with the successive letters of r (the inverse of r may also be used to build a polygon with reverse orientation). Here is the polygon associated with the simple relator  $aba^{-1}b^{-1}$  (starting at bottom left corner, counterclockwise):



Now polygons bearing (same or different) relators can be glued to each other along edges bearing the same letter. Van Kampen diagrams are the figures resulting from such connected, simply connected gluings of relator-bearing polygons. For example, the following van Kampen diagram w.r.t. the presentation  $\langle a, b | ab = ba \rangle$  (i.e. with the only relator  $aba^{-1}b^{-1}$ ) is a visual proof that if a commutes with b, then so does  $a^2$ .



An important theorem of van Kampen states that the boundary words of such planar diagrams are exactly those words equal to e in the presentation.

The connection with products of conjugates of relators is clear on the following picture, which builds the boundary word  $b^{-1}a^2ba^{-2}$  of the van Kampen diagram given above (starting at top left corner, counterclockwise) as a product  $b^{-1}rb(ab^{-1})r(ab^{-1})^{-1}$  of conjugates of the relator  $r = aba^{-1}b^{-1}$ . Shrinking this diagram (by identifying edges arising from the same point with the same label) produces the two-square one above.



Incidentally, coming back to the algorithmic decidability problems mentioned above, we see that the word problem is semi-decidable: if a word is indeed equal to e, then, searching through all van Kampen diagrams, we will eventually find one decomposing the word as a product of conjugates of relators; but proving that the word is not equal to e would a priori require examination and rejection of all possible diagrams.

#### Hyperbolic groups

A class of groups the interest for which has never declined over the years since its introduction by Gromov is that of *hyperbolic groups*. They have a combinatorial definition, *word hyperbolicity*, and a geometric one,  $\delta$ -hyperbolicity. Since (spoiler!) these two notions have turned out to be equivalent, we simply use the term hyperbolic. Word hyperbolicity. Let  $G = \langle a_1, \ldots, a_m | R \rangle$  be a group with finite presentation. Recall that any word w equal to e in G can be written as a product

$$w = \prod_{i=1}^N u_i r_i^{\pm 1} u_i^{-1}$$

(generally not in a unique way). Let N(w) be the minimal number of relators in such a writing of w. A very natural question is: How does N(w) behave when the length of w grows larger and larger?

The presentation is said to be *word-hyperbolic* if N(w) grows at most linearly with the length of w, that is, if there exists a constant C such that for any w we have  $N(w) \leq C |w|$ .

It is not difficult to see that for finite presentations, this linearity does not depend on the presentation chosen for a given group G: indeed, each generator in the second presentation is a product of (finitely many) generators in the first one, and each relator in the second presentation is a consequence of (can be written as products of conjugates of) finitely many relators in the first one, so that, since the number of generators and relators is finite, the constant C is perturbed by at most these quantities. So this yields a well-defined notion of a word-hyperbolic group.

When thought of in terms of van Kampen diagrams, this reads as follows: If w is a word equal to e in the group, then there is a van Kampen diagram D with w as its boundary word. Now by definition N(w) is the number of faces |D| of this van Kampen diagram and the length of w is the boundary length  $|\partial D|$ . So the word hyperbolicity condition rewrites as

$$|D| \leqslant C |\partial D|$$

which is a *linear isoperimetric inequality* for van Kampen diagrams. Linear isoperimetric inequalities are a negative curvature phenomenon: for example, they are satisfied by domains in the hyperbolic plane (with area standing for |D| and boundary length standing for  $|\partial D|$ ), but not by domains in the Euclidean plane (where boundary length grows only like the square root of area).

 $\delta$ -hyperbolicity. This is a more general notion defined in any *geodesic* space (that is, a metric space in which the distance between two points is realized by one or several paths between them, called *geodesic segments*). Since any graph is a geodesic space (by definition, edge-paths realize the distance), it can be applied to Cayley graphs of groups.

 $\delta$ -hyperbolicity is a way to measure how "negatively curved" or "tree-like" a space looks at large scale. In a traditional negative curvature setting, triangles have the property that the sum of their angles is less than  $2\pi$ : they are curved inwards. This is measured by Rips' condition of thinness of triangles.

A triangle in a geodesic space is specified by a triple of points (x, y, z) together with three geodesic segments joining them pairwise, noted [xy], [yz] and [zx]. For  $\delta \ge 0$ , the triangle is said to be  $\delta$ -thin if any point on [xy] lies at distance at most  $\delta$  from the union  $[yz] \cup [zx]$  of the two other sides, and similarly for points on [yz], on [zx]. That is, the "gap" at the middle of the triangle has size roughly  $\delta$ : each side does not depart too much from the other two.



For  $\delta \ge 0$ , a geodesic space is said to be  $\delta$ -hyperbolic (or simply hyperbolic if  $\delta$  is not specified) if any triangle in it is  $\delta$ -thin. Note that this must hold for all choices of geodesics between the vertices in case there are several.

The usual hyperbolic plane is hyperbolic indeed ( $\delta = 10$  works). Any bounded metric space is  $\delta$ -hyperbolic (take its diameter as  $\delta$ ). The Euclidean plane or the grid  $\mathbb{Z} \times \mathbb{Z}$  are not hyperbolic for any  $\delta \ge 0$ .

Another important example is a (finite or infinite) tree, which is 0-hyperbolic (triangles are flattened). In fact,  $\delta$ -hyperbolic spaces are those which, "seen from far away", look like trees; they can actually by approximated by trees (up to  $\delta$ ) in a very precise sense.

Now a finite group presentation is  $\delta$ -hyperbolic if the associated Cayley graph is. The most basic examples (apart from finite groups) are the free groups  $F_m$ , whose Cayley graphs w.r.t. the standard generating set are trees, hence 0-hyperbolic. Hyperbolic groups are thus a natural geometric generalization of free groups.

Importantly (but not obviously),  $\delta$ -hyperbolicity for some  $\delta \ge 0$  is preserved under quasi-isometries, although the value of  $\delta$  may change. In particular, for a group it does not depend on the choice of a generating set.

**Hyperbolicity.** A finitely presented group is word-hyperbolic if and only if it is  $\delta$ -hyperbolic for some  $\delta \ge 0$ . This is a non-trivial theorem and may actually be the shortest way to show that  $\delta$ -hyperbolicity does not depend on the presentation of a given group (though the value of  $\delta$  does).

Note that we have defined word hyperbolicity only for group presentations, whereas  $\delta$ -hyperbolicity makes sense in a more general context; but the notion of linear isoperimetric inequality of van Kampen diagrams can be extended to any geodesic space and is still equivalent to  $\delta$ -hyperbolicity.

**Small cancellation.** The small cancellation conditions are simple combinatorial criteria on a group presentation which imply hyperbolicity. Maybe the one most frequently encountered is the  $C'(\alpha)$  condition. Remember the idea of van Kampen diagrams: relators in a presentation are represented as polygons with boundary labelled by the relator. Now if two such polygons have a common subword on their boundary (more precisely, if there is a word w such that w is a subword of the boundary word of the first polygon and  $w^{-1}$  is a subword of the boundary word of the second one), we can glue them along this subword to form a two-face van Kampen diagram.

The  $C'(\alpha)$  condition for a presentation (for  $0 < \alpha < 1$ ) demands that any such gluing between two polygons bearing relators  $r_1$  and  $r_2$  in the presentation occurs along a path w of length less than  $\alpha$  times the infimum of the lengths of  $r_1$  and  $r_2$  (except for the trivial "degenerate" gluing between a polygon and the same one with inverse orientation i.e.  $r_2 = r_1^{-1}$ ).

If a group presentation satisfies the C'(1/6) condition, then the group is hyperbolic. This results from a simple Euler characteristic argument on van Kampen diagrams, which allows to show that those have enough boundary edges to ensure a linear isoperimetric inequality.

The limit case on which the significance of 1/6 is clear is the standard hexagonal tiling of the plane, which satisfies  $C'(1/6 + \varepsilon)$  for any  $\varepsilon > 0$  but not C'(1/6), and which is not hyperbolic (compare the heptagonal tiling of the hyperbolic plane).

All in one: an example. Let G be the group presented by

$$\langle a, b, c, d \mid aba^{-1}b^{-1}cdc^{-1}d^{-1} = e \rangle$$

which means that the only polygon appearing in van Kampen diagrams is the following octogon.



Now the only ways to glue such a polygon with a copy of itself consist in gluing the two *a*'s, or gluing the two *b*'s, or the two *c*'s, or the two *d*'s, but there is no way to find a gluing along two consecutive letters. Since the length of the relator is 8, this means that this presentation satisfies the  $C'(1/8 + \varepsilon)$  small cancellation condition (but not C'(1/8)). Since 1/8 < 1/6 the criterion above applies and so this group is hyperbolic.

This example is not anecdotic. Copies of the above octogon can be glued (in the allowed ways: an a with an a, etc.) to form the standard octogonal tiling of the hyperbolic plane. Actually this tiling is exactly the Cayley graph of the group, which we represent on pages 13 and 14 in two different Euclidean views: a Poincaré model centered either on a face, emphasizing the tiling aspect, or on an element, emphasizing the Cayley graph. (When looking at the figures, keep in mind that the hyperbolic metric goes to infinity close to the disk boundary, so that all octogons are actually isometric and play the same role. The second picture is obtained from the first one by performing a hyperbolic isometry sending the left vertex of the large bottom a to the center of the disk.)

Since the hyperbolic plane is hyperbolic, this is another way to check hyperbolicity of this group: the group identifies with vertices of the tiling, equipped with the edge metric, which is a discrete subset quasi-isometric to the whole hyperbolic plane. Note how tree-like the Cayley graph looks in spite of the presence of cycles of length 8.

A group naturally acts on its Cayley graph by left multiplication and so we can take the quotient of this hyperbolic tiling by action of the group. This provides a two-dimensional object, the fundamental group of which is the group.

Let us take a closer look at this quotient. A single octogon is a fundamental domain for the action of the group on the tiling (i.e. the set of all translates of some octogon by the group exactly produces the tiling) and so the quotient is obtained by taking a single octogon and identifying its edges according to the labels. So take a single copy of the octogon above and twist it in three-dimensional space so that the two edges labelled by a are identified (preserving the orientation implied by the arrows); then identify the two edges labelled by b, then the two ones labelled by c, then the ones labelled by d (all eight vertices will be identified along the way). This actually leaves us with a surface of genus two, i.e. the gluing of two tori (this is quite hard to figure out—check it first in the simpler case of a square  $aba^{-1}b^{-1}$  instead of the octogon above: this square transforms into a torus).

So the group can be represented as the fundamental group of a surface of genus 2. This surface inherits a metric of negative curvature from that of the hyperbolic plane. Actually, fundamental groups of negatively curved compact surfaces, or of polyhedra with negative curvature in some combinatorial sense, were an important motivation for the introduction of hyperbolic groups and initially the main source of examples.





# Suggested reading

Groups as geometric objects, hyperbolic groups, quasi-isometries, Cayley graphs... É. Ghys, P. de la Harpe, Sur les groupes hyperboliques d'après Mikhael Gromov, Progress in Math. 83, Birkhäuser (1990).

Groups as geometric objects, quasi-isometries...

M. Gromov, Infinite groups as geometric objects, in Proceedings of the International Congress of Mathematicians, Vol. 1, 2, Warsaw (1983), 385–392, PWN, Warsaw, 1984.

Group presentations, free groups, van Kampen diagrams (under the name "cancellation diagrams"), decision problems...

J. J. Rotman, An introduction to the theory of groups, fourth edition, Graduate Texts in Mathematics 148, Springer (1995), especially chapters 11 and 12.

Group presentations, free groups, van Kampen diagrams, small cancellation theory...

R. C. Lyndon, P. E. Schupp, *Combinatorial group theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete **89**, Springer (1977).

Negative curvature, hyperbolic groups, isoperimetric inequalities...

M. R. Bridson, A. Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der mathematischen Wissenschaften **319**, Springer (1999), especially chapters H and  $\Gamma$ .