Spectral interpretations of property (T)

Yann Ollivier

November 2003

In this document we describe the spectral characterization of property (T), that is, its interpretation in terms of group-equivariant random walks. This leads to some efficient sufficient criteria for property (T), which can, notably, be algorithmically checked on a given group presentation. This is but one of the numerous lines of recent progress of research on property (T) (see [V]).

Credit for the results presented here is to be shared between Garland, Pansu, Ballmann, Swiatkowski, Żuk, Gromov. The presentation is much inspired by Étienne Ghys (see [Gh]).

We assume that the reader is already familiar with property (T). If not, the only definition we will use is the following: A finitely generated group G has property (T) if and only if for any finite generating set S of G, there exists a constant κ (Kazhdan constant) such that for any unitary representation π of G in an Hilbert space $\mathcal{H}$, either there is an invariant vector in $\mathcal{H}$ for the action of G, or, for any $u \in \mathcal{H}$ of norm 1, there exists some $s \in S$ such that $\|\pi(s)u - u\| \geqslant \kappa$. That is, if every vector is displaced, then we know what the minimal displacement is. Checking this property for one generating set S is enough to ensure property (T).

In the following *all groups are assumed to be finitely generated.*

1 Invariant random walks

We begin by giving some definitions about G-invariant random walks on discrete sets and their spectral properties.

Random walks on discrete sets. Usually, a discrete random walk (or Markov chain) is given by a set X together with, for each $x \in X$, a probability measure $\mu(x \to \cdot)$ so that $\mu(x \to y)$ represents the transition probability from x to y.

We want to study an invariant analogue of this notion. As we will work with finitely generated groups we make further finiteness assumptions.

Definition 1 (G-invariant random walk) – *Let G be a discrete group. A G-invariant random walk is a random walk $\left(X, (\mu(x \to \cdot))_{x\in X}\right)$ such that:*

- *X is enumerable and X/G is finite;*
- *for each $x \in X$, there are only a finite number of $y \in X$ such that $\mu(x \to y) > 0$;*
- *μ is G-invariant, that is, for all $g \in G$, for all $x, y \in X$, we have $\mu(gx \to gy) = \mu(x \to y)$;*
- *the graph on X whose edges consist in pairs $(x, y) \in X \times X$ such that $\mu(x \to y) > 0$ is connected as an oriented graph.*

If moreover G acts freely on X, the random walk is said to be free.

The connectedness assumption discards some degenerate cases.

In particular, this allows to define a random walk $\bar{\mu}$ on X/G, by setting $\bar{\mu}(\bar{x} \to \bar{y}) = \sum_y \mu(x \to y)$ where x is some preimage of $\bar{x}$ and y runs through all preimages of $\bar{y}$. Note that by the finiteness assumption on the supports of $\mu(x \to \cdot)$, the sum is finite. Invariance ensures that this does not depend on the choice of x. This we will call the *quotient random walk*.

If we are given two G-invariant random walks (X, μ_1) and (X, μ_2) on the same G-set X, then we can define their *convolution product* (or simply *product*) by

$$\mu_1 * \mu_2 \,(x \to y) = \sum_{z \in X} \mu_1(x \to z)\, \mu_2(z \to y)$$

for all $x, y \in X$. (This sum is always finite.) It is immediate to check that this is a G-invariant random walk as well. In particular we can define the n-th iterate $\mu^{*n} = \mu * \cdots * \mu$ (n times) of μ. Of course this corresponds to performing n steps of the random walk given by μ.

Let us give the most standard example.

Example 2 – *Suppose that G is generated by some finite set S. Then the random walk on G itself defined by $\mu(x \to y) = \frac{1}{|S|}$ if $y = xs$ for some $s \in S$, and 0 otherwise, is a G-invariant random walk, which we call the* random walk on G arising from S. *It can naturally be seen as the simple random walk in the Cayley graph of G with generating set S.*

Now it is well-known in group theory that dealing with *symmetric* generating sets (that is, $s^{-1} \in S$ when $s \in S$) is much easier, since for example the Cayley graph is unoriented in this case. In our framework the analogous definition is the following.

Definition 3 (Symmetric random walk) – *Let (X, μ) be an G-invariant random walk. It is said to be* symmetric with respect to ν *if ν is a non-zero G-invariant measure on X such that*

$$\nu(x)\mu(x \to y) = \nu(y)\mu(y \to x)$$

for all $x, y \in X$. It is said to be symmetric *if there exists such a non-zero measure ν.*

In this situation, it is clear that ν is a stationary measure for the random walk given by μ (recall a measure is called stationary if $\nu(x) = \sum \mu(y \to x)\nu(y)$). In fact symmetry is stronger: not only the weights of each point are stable under the random walk, but for each pair of points, the mass exchanged between them is zero. The symmetric quantity $\nu(x)\mu(x \to y)$ can be seen as a measure on the unoriented edges of the graph of points of X joined by μ.

Once again, the most natural situation is when X is G itself and the random walk is the simple random walk with respect to a *symmetric* set of generators. Let us, however, give a second example.

Example 4 – *Let $\bar{X}$ be a finite connected unoriented complex with fundamental group G and let X be the universal cover of $\bar{X}$. For x in X let $\nu(x)$ be the number of edges originating from x, and let $\mu(x \to y)$ be $\frac{1}{\nu(x)}$ times the number of edges from x to y. Then (X, μ) is a symmetric free G-invariant random walk with respect to ν, which we call the* natural random walk *on X.*

Note that this construction only depends on the 1-skeleton of X.

This example shows that, even in the case of a free G-invariant random walk (when, as a G-set, X is isomorphic to $G \times X/G$), the G-invariant random walk can convey some homological information in the ways that the random walk wheels around G/X with a non-trivial action on the fibers.

Spectral properties of random walks. The speed of convergence of a symmetric random walk to its stationary measure is controlled by spectral quantities, namely by the eigenvalues of the averaging operator on ℓ^2 functions on the space. Here we have a richer structure given by the G-action. Namely, instead of considering all functions on our space X, we may decide to keep only *equivariant* functions with respect to some representation of G. So we view the G-equivariant random walk as a random walk on the quotient, with some richer structure in the fibers.

Let $\mathcal{H}$ be a Hilbert space and let π be a unitary linear representation of the group G in $\mathcal{H}$. Let (X, μ) be a G-invariant symmetric random walk on X with respect to the measure ν. A function $f : X \to \mathcal{H}$ is said to be *G-equivariant* if for any $x \in X$ and $g \in G$ we have $f(gx) = \pi(g)f(x)$.

The usual situation when we deal with a random walk on a finite graph $\bar{X}$ is simply the case $\mathcal{H} = \mathbb{C}$ with the trivial action on $\mathcal{H}$ and with $X = \bar{X} \times G$.

Let $\mathcal{E}_\pi$ be the space of equivariant functions of X to $\mathcal{H}$ (which may be $\{0\}$ if G does not act freely enough). As G acts unitarily on $\mathcal{H}$, the function $\nu(x) \left\| f(x) \right\|^2$ is invariant on X, and thus it is meaningful to sum it on X/G. This endows $\mathcal{E}_\pi$ with a Hilbertian structure by setting

$$\left\| f \right\|_\nu^2 = \sum_{x \in X/G} \left\| f(x) \right\|^2 \nu(x) = \int_{X/G} \left\| f \right\|^2 \mathrm{d}\nu$$

and of course

$$\langle f \mid g \rangle_\nu = \sum_{x \in X/G} \langle f(x) \mid g(x) \rangle \, \nu(x) = \int_{X/G} \langle f \mid g \rangle \, \mathrm{d}\nu$$

for each $f, g \in \mathcal{E}_\pi$.

In usual random walks on a graph there is an averaging operator on functions on the graph (also equal to 1 minus the discrete Laplacian), whose spectrum controls the speed of convergence to the possible stationary measure. We give analogous definitions here.

Let $f \in \mathcal{E}_\pi$ be an equivariant function on X. The *averaging operator* M is defined by

$$Mf(x) = \sum_{y \in X} \mu(x \to y) f(y)$$

for every $x \in X$. (Once again the sum is finite.) With obvious notations, we have $M_{\mu_1 * \mu_2} = M_{\mu_1} M_{\mu_2}$.

It is important to note but easy to check, using symmetry of μ, that M is a symmetric operator for the Hilbertian structure on $\mathcal{E}_\pi$ defined above. In particular, we can define the spectrum of M. This spectrum is contained in $[-1; 1]$.

Of course, if f is constant then it is stable by M. So for example if G acts trivially on $\mathcal{H}$, then 1 will lie in the spectrum of M. Similarly, for any group G there exists an invariant random walk for which -1 is an eigenvalue of the averaging operator. Indeed, take $X = \{0, 1\} \times G$ with G acting on G by left multiplication. Let S be some finite generating set of G, such that

$e \in S$. For $x = (a, g) \in \{0,1\} \times G$, define $\mu(x \to y)$ as $1/|S|$ if y is of the form $(1-a, gs)$ for some $s \in S$, and 0 otherwise. This is a G-invariant random walk, and its quotient on $\{0,1\}$ is the Markov chain which exchanges 0 and 1 at each step. (We include e in S to ensure connectedness.)

We finally arrive at the crux of the thing.

Definition 5 (G-invariant random walk with spectral gap) – *Let G be a discrete group and let (X, μ) be a G-invariant random walk which is symmetric with respect to the measure ν. We say that this random walk* has a spectral gap *if there exists a number $\sigma < 1$ such that for any unitary representation π of G into a Hilbert space $\mathcal{H}$, the spectrum of the averaging operator M acting on $\mathcal{E}_\pi$ is included in the set $[-1; \sigma] \cup \{1\}$, where $\mathcal{E}_\pi$ is the space of all equivariant functions from X to $\mathcal{H}$, with the Hilbertian structure above.*

The quantity $1 - \sigma$ is called the spectral gap.

If, in the same circumstances, the spectrum of M is included in the set $\{-1\} \cup [-\sigma; \sigma] \cup \{1\}$ then we say that this random walk has a double-sided spectral gap.

Of course it is equivalent to ask that, for any representation, there exists a spectral gap. Indeed, if there is a sequence of representations such that the spectral gap tends to 0, then the direct orthogonal sum of these has no spectral gap. So if for any representation there is a spectral gap, then the spectral gap can be taken uniform on all representations.

G-invariant random walks versus ordinary random walks. Let us give two examples of application of this definition.

If, in the definition of a G-invariant random walk with spectral gap, we restrict ourselves to $\mathcal{H} = \mathbb{C}$ with the trivial representation, then we get back the definition of the usual spectral gap of the quotient random walk on X/G. Hence we get the following.

Proposition 6 – *Let (X, μ) be a G-invariant symmetric random walk with spectral gap λ. Then the spectral gap of the quotient random walk on X/G is at least λ. The same holds for double-sided spectral gap.*

In another direction, if we are given a set X with an action of G, we can define an ordinary (non G-invariant) random walk on X with respect to some symmetric generating set S of G by deciding that at each step, we pick some $s \in S$ and we let it act on X. The spectral gap of this ordinary random walk is controlled by the random walk on G with respect to this generating set, using the representation of G in $\ell^2(X)$.

Proposition 7 – *Let G be a discrete group generated by a finite symmetric set S. Let X be a set on which G acts transitively, and define the symmetric ordinary random walk μ on X by setting*

$$\mu(x \to y) = \frac{1}{|S|} \left| \{ s \in S, sx = y \} \right|$$

for all $x, y \in X$. Suppose that the G-invariant random walk on G arising from S has spectral gap λ. Then the spectral gap of the ordinary random walk (X, μ) is at least λ.

Proof – Let (G, μ_G) be the symmetric G-invariant random walk on G arising from the generating set S, which is symmetric with respect to the counting measure ν_G. Let π be the action of G on $\ell^2(X)$ given by $\pi(g)(f)(x) = f(gx)$ for $g \in G$, $f \in \ell^2(X)$, $x \in X$. For $f \in \ell^2(X)$, let $\bar{f}$ be the function of $\mathcal{E}_\pi$ defined by $\bar{f}(g) = \pi(g) f$ for $g \in G$. We have $\langle \bar{f} \mid \bar{g} \rangle_{\nu_G} = \langle \bar{f}(e) \mid \bar{g}(e) \rangle_{\ell^2(X)} = \langle f \mid g \rangle_{\ell^2(X)}$.

For $x \in X$ we have

$$\left(\left(M\bar{f} \right) e \right)(x) = \frac{1}{|S|} \sum_{s \in S} \bar{f}(s)(x) = \frac{1}{|S|} \sum_{s \in S} f(sx)$$

and on the other hand

$$\overline{Mf}(e)(x) = (Mf)(x) = \frac{1}{|S|} \sum_{s \in S} f(sx)$$

so that $\left\langle \bar{f} \mid \overline{Mf} \right\rangle_{\nu_G} = \langle f \mid Mf \rangle_{\ell^2(X)}$. So the spectra of M acting on $\mathcal{E}_\pi$ and M acting on $\ell^2(X)$ coincide. □

Criteria for the spectral gap. For random walks on finite graphs, it is well-known that the spectral radius of the averaging operator controls the norm of the gradient of a function (seen as a function of the edges) in terms of the norm of the function.

If f is an equivariant function on X, we can define its gradient $\mathrm{d}f$ on X^2 by $\mathrm{d}f(x, y) = f(y) - f(x)$. This will be an equivariant function again (for the diagonal action on X^2). We have two natural symmetric measures on X^2, namely $\nu(x)\nu(y)$ and $\mu(x, y)\nu(x)$, but a priori only the latter has a finite mass on X^2/G (however see Proposition 9 for the case when ν has finite mass). So we will define norms with respect to this latter measure, which can be seen as a measure on the edges of the graph with vertices in X, edges being defined by positivity of $\mu(x \to y)$.

So let $F : X^2 \to \mathcal{H}$ be a G-equivariant function on X^2, that is, satisfying $F(gx, gy) = \pi(g)F(x, y)$ for all $g \in G$, $x, y \in X$. We set

$$\|F\|_\mu^2 = \sum_{(x,y) \in X^2/G} \|F(x, y)\|^2 \; \mu(x \to y)\nu(x)$$

which is meaningful since the summand is G-invariant. Note that the values of F on couples not connected by μ are not taken into account. We define $\langle F \mid F' \rangle_\mu$ accordingly for G-equivariant functions on X^2.

Now if f is an equivariant function on X, we can define its *energy* as

$$E_\mu(f) = \frac{1}{2} \|\mathrm{d}f\|_\mu^2 = \langle f \mid (\mathrm{Id} - M) f \rangle_\nu$$

where we put a factor $1/2$ since each difference $\|f(x) - f(y)\|^2$ is counted once with $\nu(x)\mu(x \to y)$ and once with $\nu(y)\mu(y \to x)$ (which are equal since μ is symmetric). If f is not constant this is non-zero.

This quantity evaluates the differences of the values of f on points connected by one step of the random walk. A natural question is: How is this related to the differences of the values of f on points connected by k steps of a random walk? We would expect these variations to be roughly k times greater. Indeed: since $E_{\mu^{*k}}(f) = \left\langle f \mid (\mathrm{Id} - M^k) f \right\rangle_\nu$, one gets (using the spectral decomposition for M) that for any representation π, for any integer $k \geqslant 1$, there exists a constant $c_k < k$ such that for any f in $\mathcal{E}_\pi$ we have $E_{\mu^{*k}}(f) \leqslant c_k E_\mu(f)$. Actually, the constant c_k is given by $c_k = (1 - \sigma^k)/(1 - \sigma) < k$ where σ is the greatest eigenvalue (apart from 1) of M acting on $\mathcal{E}_\pi$.

Conversely, as $(1 - \sigma^k)/(1 - \sigma) < k$ implies $\sigma < 1$, we get that in case c_k does not depend on the representation π, then the random walk μ has a spectral gap.

Proposition 8 – *Let (X, μ) be a G-equivariant random walk. The following are equivalent:*

(i) (X, μ) *has a spectral gap;*

(ii) *for any integer* $k \geqslant 2$*, there exists a constant* $c_k < k$ *such that for any unitary representation* π *of* G*, for any* $f \in \mathcal{E}_\pi$ *we have* $E_{\mu^{*k}}(f) \leqslant c_k E_\mu(f)$*;*

(iii) *there exists an integer* $k \geqslant 2$ *and a constant* $c_k < k$ *such that for any unitary representation* π *of* G*, for any* $f \in \mathcal{E}_\pi$ *we have* $E_{\mu^{*k}}(f) \leqslant c_k E_\mu(f)$*.*

Moreover in this case, the largest eigenvalue σ *(apart from* 1*) and the constants* c_k *are linked by the relation* $c_k = 1 + \sigma + \cdots + \sigma^{k-1} \leqslant 1/(1-\sigma)$*.*

Let us switch back for some time to ordinary random walks. Let μ be an ordinary random walk on the finite set X, symmetric with respect to measure ν. The energy $E_\mu(f)$ is the average square variation of f on points joined by one step of the random walk μ. This could be compared to the average square variation of f on all pairs of points, that is, to the variance of f defined as

$$\operatorname{Var} f = \frac{1}{2\nu(X)} \sum_{x,y \in X} \|f(x) - f(y)\|^2 \, \nu(x) \, \nu(y) = \|f - \mathbb{E}f\|_\nu^2$$

where $\mathbb{E}f$ stands for the average of f under measure ν, which is well-defined since X is finite. This is the ν-norm of the projection of f on the orthogonal of constant functions.

Let M be the averaging operator associated to μ and let σ be its greatest eigenvalue, apart from 1. Since the eigenspace associated to the eigenvalue 1 of M is the space of constant functions, for f a function of average 0 we get $\langle Mf \mid f \rangle_\mu \leqslant \sigma \|f\|_\nu^2$. But, as we saw above, we have $E_\mu(f) = \langle f \mid (1-M)f \rangle_\nu$ and so $E_\mu(f) \geqslant (1-\sigma) \|f\|_\nu^2$. So we have shown the following.

Proposition 9 – *Let* (X, μ) *be an ordinary random walk, symmetric with respect to measure* ν*. Suppose that this random walk has spectral gap* λ*. Then*

$$\operatorname{Var} f \leqslant \frac{1}{\lambda} E_\mu(f)$$

for any function f *on* X *with values in a Hilbert space.*

In other words, the spectral gap controls the maximum ratio between the average square variation of f on points joined by one step of the random walk, and the average square variation of f on all pairs of points. This can also serve as a definition of the spectral gap (as the optimal constant in this inequality). This kind of inequality is usually termed a *Poincaré inequality* in analysis (cf. [?]).

This inequality is valid for G-invariant random walks as well, provided $\operatorname{Var} f$ is defined as the squared ν-norm of the projection of f to the orthogonal of the constants.

Last, let us mention a common trick used when a double-sided spectral gap is needed where only a spectral gap is known. For any random walk μ we can define the random walk μ' by deciding that at each step, with probability 1/2 we do not move and with probability 1/2, we perform the random walk μ. The associated averaging operator is $M_{\mu'} = (1 + M_\mu)/2$, which has non-negative spectrum. This is termed *lazy random walk* and we will use it crucially below.

2 The spectral characterization of property (T)

Let us now state the theorem motivating all the above definitions.

Theorem 10 – *Let G be a discrete group. The following are equivalent:*

(i) G has property (T);

(ii) any G-invariant symmetric random walk has a double-sided spectral gap;

(iii) there exists a free G-invariant symmetric random walk with spectral gap.

Some lemmas. Before proceeding to the proof, we need to recall some basic facts on ordinary random walks and geometry in Hilbert spaces.

Consider an ordinary random walk μ on a finite set X, which is symmetric w.r.t. some measure ν. We can associate to it, as above, an averaging operator M on the space of all complex functions on X. Then, it is well-known that there exists a number $\sigma < 1$ such that for any complex-valued function f on X such that f is of zero mean ("orthogonal to the constants"), we have $\langle Mf \mid f \rangle_\nu \leqslant \sigma \|f\|_\nu^2$.

This immediately extends (with the same constant) to functions with values in some Hilbert space $\mathcal{H}$, simply by decomposing in a Hilbertian basis. Namely:

Lemma 11 – *Let μ be an ordinary random walk on a finite set X, symmetric with respect to measure ν. Let M be the associated averaging operator. Then there exists a constant $\sigma < 1$ such that, for any Hilbert space $\mathcal{H}$, for any function $f : X \to \mathcal{H}$ which is orthogonal to the constant functions, we have $\langle Mf \mid f \rangle_\nu \leqslant \sigma \|f\|_\nu^2$.*

Second, we need a lemma about the norms of averages in Hilbert spaces.

Lemma 12 – *Let u and $u_1, \dots, u_k$ be vectors in a Hilbert space, all of them of norm 1. Let $a_1, \dots, a_k$ be positive numbers such that $\sum a_i = 1$. Suppose that there exists some i_0 such that $\|u - u_{i_0}\| \geqslant \varepsilon$. Then $\mathrm{Re} \langle u \mid \sum a_i u_i \rangle \leqslant 1 - a_{i_0}\varepsilon^2/2$, where Re denotes the real part.*

Proof of the lemma – Using $\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\mathrm{Re} \langle x \mid y \rangle$, we get $\mathrm{Re} \langle u_{i_0} \mid u \rangle \leqslant 1 - \varepsilon^2/2$.

Since all u_i's are of norm 1, we have $\mathrm{Re} \langle u_i \mid u \rangle \leqslant 1$ for any i. Thus, we have $\sum_i a_i \mathrm{Re} \langle u_i \mid u \rangle \leqslant 1 - a_{i_0}\varepsilon^2/2$. □

The squaring is clear on a picture: in the Euclidean plane, take u a unitary vector and u_1, u_2 two vectors on each side of u making the same small angle ε with it. Then $\|u - (u_1 + u_2)/2\|$ is of order $\varepsilon^2/2$.

Our next lemma is a simple exercise in using the arithmetico-geometric inequality $2ab \leqslant a^2 + b^2$.

Lemma 13 – *Let $(\nu_i)_{i \in I}$ be a family of non-negative numbers. Let $(\mu_{ij})_{i,j \in I^2}$ be a family of non-negative numbers such that for any i we have $\sum_j \mu_{ij} = 1$. Assume furthermore that $\nu_i \mu_{ij} = \nu_j \mu_{ji}$ for any i, j. Let x_i be a family of real numbers. Then $\sum_{ij} \nu_i \mu_{ij} x_i x_j \leqslant \sum_i \nu_i x_i^2$.*

Proof of Theorem 10 –

$(i) \Rightarrow (ii)$**.** Let G be a discrete group and let (X, μ) be a G-invariant symmetric random walk with respect to measure ν.

We begin with spectral gap-ness instead of double-sided spectral gap-ness. We have to prove that the averaging operator of this random walk has a spectrum included in $[-1; \sigma] \cup \{1\}$ for some $\sigma < 1$.

Let S be a finite symmetric generating set of G. Let κ be a Kazhdan constant of G with respect to S, that is, for any unitary representation π in the Hilbert space $\mathcal{H}$ without invariant vectors, for any $u \in \mathcal{H}$ of norm 1, there exists $s \in S$ such that $\|\pi(s)u - u\| \geqslant \kappa$.

Let p be the quotient map $X \to X/G$. For $i \in X/G$ let $X_i = p^{-1}(i)$ and choose once and for all some base point $x_i \in X_i$.

By assumption the graph on X whose edges are the pairs (x, y) with $\mu(x \to y) > 0$ is connected. For each $i \in X/G$ and $s \in S$ we can thus find an integer $k_{i,s}$ such that the $k_{i,s}$-step transition probability $\mu^{*k_{i,s}}(x_i \to sx_i)$ is positive.

Now take $k = \max_{i \in X/G, s \in S} k_{i,s}$. Consider the lazy random walk μ' associated to μ. Since at each step this random walk has positive probability to stay in place, we have $\mu'^{*k}(x_i \to sx_i) > 0$ for any $i \in X/G$, $s \in S$. Denote by $\lambda = \mu'^{*k}$ this random walk. Also set $\alpha = \min_{s \in S} \min_{i \in X/G} \lambda(x_i \to sx_i)$. By definition of λ this is positive.

Let π be a unitary representation of G on the Hilbert space $\mathcal{H}$. Let $f \in \mathcal{E}_\pi$ be a G-equivariant function on X which is orthogonal to the constants in $\mathcal{E}_\pi$.

Let M be the averaging operator on $\mathcal{E}_\pi$ associated to λ. Since $\lambda = (\mu/2 + 1/2)^{*k}$, proving a spectral gap for λ or μ is equivalent.

We want to show that there exists a constant $\sigma < 1$, independent of the representation π, such that for any $f \in \mathcal{E}_\pi$ orthogonal to the constants we have $\langle Mf \mid f \rangle_\nu \leqslant \sigma \|f\|_\nu^2$.

Let $\mathcal{F}_1$ be the subspace of $\mathcal{E}_\pi$ made of functions of X to $\mathcal{H}$ that are constant on each orbit X_i (this may well be $\{0\}$). Since functions in $\mathcal{E}_\pi$ are equivariant, if $f \in \mathcal{F}_1$ then for each $x \in X$, the vector $f(x) \in \mathcal{H}$ is invariant under the action of G on $\mathcal{H}$. So $\mathcal{F}_1$ is made of the functions on X which, on each orbit X_i, are equal to some vector $v_i \in \mathcal{H}$ fixed under the action of G on $\mathcal{H}$.

Let $\mathcal{F}_2$ be the orthogonal of $\mathcal{F}_1$ in $\mathcal{E}_\pi$. Let $f \in \mathcal{E}_\pi$. Decompose $f = f_1 + f_2$ with $f_1 \in \mathcal{F}_1$, $f_2 \in \mathcal{F}_2$.

Now f_1 is constant on each component X_i. So we can view f_1 as a function $\bar{f}_1$ on X/G with values in $\mathcal{H}$. Let $\bar{\lambda}$ be the quotient random walk on X/G. It is an ordinary random walk, which is symmetric with respect to the measure $\bar{\nu}$. Let $\bar{M}$ be the associated averaging operator on X/G by π.

Let σ_1 be the spectral gap of $\bar{\lambda}$. Given that f_1 is constant on each component X_i, unwinding the definitions, one almost tautologically checks that

$$\langle Mf_1 \mid f_1 \rangle_\nu = \langle \bar{M}\bar{f}_1 \mid \bar{f}_1 \rangle_{\bar{\nu}}$$

and so by Lemma 12 (spectral gap in X/G) we get

$$\langle Mf_1 \mid f_1 \rangle_\nu \leqslant \sigma_1 \left\|\bar{f}_1\right\|_{\bar{\nu}}^2 = \sigma_1 \|f_1\|_\nu^2$$

which holds since $\bar{f}_1$ is orthogonal to the constants. This is the first and simplest half of the job.

Now for the f_2 component. It is time to use property (T). We want to show that

$$\langle Mf_2 \mid f_2 \rangle_\nu \leqslant \sigma_2 \|f_2\|_\nu^2$$

for some constant $\sigma_2 < 1$ independent of π.

By definition we have (Re denoting the real part)

$$\begin{aligned}
\langle Mf_2 \mid f_2 \rangle_\nu &= \sum_i \mathrm{Re} \left\langle \sum_{y \in X} \lambda(x_i \to y) f_2(y) \mid f_2(x_i) \right\rangle \nu(x_i) \\
&= \sum_i \mathrm{Re} \left\langle \sum_{y \in X_i} \lambda(x_i \to y) f_2(y) \mid f_2(x_i) \right\rangle \nu(x_i) + \\
&\quad \sum_i \sum_{y \notin X_i} \lambda(x_i \to y)\, \mathrm{Re} \langle f_2(y) \mid f_2(x_i) \rangle \, \nu(x_i) \\
&\leqslant \sum_i \mathrm{Re} \left\langle \sum_{y \in X_i} \lambda(x_i \to y) f_2(y) \mid f_2(x_i) \right\rangle \nu(x_i) + \\
&\quad \sum_i \sum_{j \neq i} \bar{\lambda}(i \to j) \, \|f_2(x_i)\| \, \|f_2(x_j)\| \, \nu(x_i)
\end{aligned}$$

since on each X_i, the norm of f_2 is constant.

Now fix i and consider the first term. Consider the space of functions $f \in \mathcal{F}_2$ restricted to X_i: these functions take values in $\mathcal{H}$ and more precisely, by definition of $\mathcal{F}_2$, in the orthogonal in $\mathcal{H}$ of the invariant vectors for the action of G on $\mathcal{H}$. So we can apply property (T) and the Kazhdan constant given above: we have

$$\|f_2(x_i) - \pi(s_i) f_2(x_i)\| \geqslant \kappa \, \|f_2(x_i)\|$$

for some $s_i \in S$.

But by equivariance, $\pi(s_i) f_2(x_i)$ is equal to $f_2(s_i x_i)$. But by construction of λ, we have $\lambda(x_i \to s x_i) > 0$. Now apply Lemma 12 (after some renormalizations) and get

$$\mathrm{Re} \left\langle \sum_{y \in X_i} \lambda(x_i \to y) f_2(y) \mid f_2(x_i) \right\rangle \leqslant \left(\bar{\lambda}(i \to i) - \lambda(x_i \to s_i x_i) \frac{\kappa^2}{2} \right) \|f_2(x_i)\|^2$$

so that, reminding that $\alpha = \min_{s \in S} \min_{i \in X/G} \lambda(x_i \to s x_i) > 0$, we get from the above

$$\begin{aligned}
\langle Mf_2 \mid f_2 \rangle_\nu \leqslant\; & \sum_i \bar{\lambda}(i \to i) \, \|f_2(x_i)\|^2 \, \nu(x_i) - \frac{\alpha \kappa^2}{2} \sum_i \|f_2(x_i)\|^2 \, \nu(x_i) + \\
& \sum_i \sum_{j \neq i} \bar{\lambda}(i \to j) \, \|f_2(x_i)\| \, \|f_2(x_j)\| \, \nu(x_i)
\end{aligned}$$

where it is important to remind that κ and α are defined independently of the representation π at play.

Now applying Lemma 13 to the first and last terms leads to

$$\langle Mf_2 \mid f_2 \rangle_\nu \leqslant \sum_i \nu(x_i) \, \|f(x_i)\|^2 - \frac{\alpha \kappa^2}{2} \sum_i \nu(x_i) \, \|f(x_i)\|^2$$

hence the conclusion since by definition $\sum \nu(x_i) \|f_2(x_i)\|^2 = \|f_2\|_\nu^2$.

So finally we can set $\sigma = \max(\sigma_1, \alpha\kappa^2/2)$ where σ_1 comes from the random walk on X/G, κ is a Kazhdan constant and α depends on the structure of the G-equivariant random walk; all of this being independent of the representation π (but nevertheless, the decomposition $f = f_1 + f_2$ actually depends on π).

This does not refer to the original random walk μ but to some iterate of the associated lazy random walk, but as we said above, bounding the spectrum of the latter also bounds that of the former.

The "σ_1" part accounts for the structure of the random walk on X/G. It is the only one present, for example, if G acts trivially on X. The "κ^2" or "Kazhdan" part amounts for what is going on with the G-action. This is the only one present, for example, if G acts transitively (so that X/G is only one point).

Now for double-sided spectral gap-ness. The idea is to apply the above to the random walk μ^{*2}, the spectrum of which is non-negative.

But the graph of points of X which can be connected by an even number of steps of μ need not be connected, whereas we crucially used connectedness above. If this graph is connected, then applying the above to μ^{*2} immediately provides the desired evaluation of the negative part of the spectrum for μ.

Now suppose this graph is not connected. Let x_0 be any point of X; let C_0 be the set of points of X which can be reached from x_0 in an even number of steps of the random walk μ, and let $C_1 = X \setminus C_0$. Using G-invariance, it is easy to see that, for any $g \in G$ acting on X, either g stabilizes C_0 and C_1 or g exchanges them. This gives rise to a morphism ε from G to $\{-1, 1\}$.

Thus, for any representation π we can define another one $\tilde{\pi}$ by twisting it by ε. The space $\mathcal{E}_{\tilde{\pi}}$ is obtained from $\mathcal{E}_\pi$ by changing the sign of the values of equivariant functions on C_1. Since one step of the random walk μ only connects points between C_0 and C_1, it is easy to see that the spectrum of the averaging operator on $\mathcal{E}_{\tilde{\pi}}$ is exactly the opposite of that on $\mathcal{E}_\pi$. So in order to control the negative part of the spectrum for the representation π, it is enough to apply the above to the representation $\tilde{\pi}$.

This proves the first (and most delicate) implication of the theorem. We will see below that when the random walk μ arises from some generating set of G, things become somewhat simpler since we do not have to take some iterate or to decompose into different components.

(*ii*) ⇒ (*iii*). Clear (there exist some free G-invariant random walks!).

(*iii*) ⇒ (*i*). Let (X, μ) be a G-invariant random walk, symmetric with respect to measure ν, such that G acts freely on X. Let σ be such that the spectrum of the averaging operator M of μ on any representation is included in $[-1; \sigma] \cup \{1\}$.

Let as above p be the projection from X to X/G and, for each $i \in X/G$, set $X_i = p^{-1}(i)$ and choose some $x_i \in X_i$.

As G acts freely on X, if x and y both belong to X_i the meaning of yx^{-1} is a well-defined element of G. Let

$$S = \{yx_j^{-1}, j \in X/G, y \in X_j, \exists i \in X/G, \mu(x_i \to y) > 0\}$$

be a set of elements of G with the following property: if $\mu(x_i \to y) > 0$, then $y = sx_j$ for some $s \in S$, $j \in X/G$. By the finiteness assumption in the definition of G-invariant random walks, this set is finite.

Let π be a representation of G on the Hilbert space $\mathcal{H}$ without invariant vectors. As 1 is isolated in the spectrum of M it has to be an eigenvalue. A fixed point of M in $\mathcal{E}_\pi$ is, by connectedness of the graph of the random walk, a constant vector on X; but as there are no invariant vectors in the representation π, there is no non-zero constant equivariant function on X. So there is no eigenvector of M in $\mathcal{E}_\pi$ associated to 1. Hence, the spectrum of M on $\mathcal{E}_\pi$ is included in $[-1;\sigma]$.

As G acts freely on X, if for each $i \in X/G$ we choose a vector $v_i \in \mathcal{H}$ then this determines a unique equivariant function on X by imposing $f(x_i) = v_i$.

For some $\varepsilon > 0$, let $v \in \mathcal{H}$ be a (S,ε)-invariant vector for the action of G. Define f in $\mathcal{E}_\pi$ by setting $f(x_i) = v$ for each $i \in X/G$ and extending this definition equivariantly to X. Now, by definition of S, for any $i \in X/G$, for any $y \in X$ such that $\mu(x_i \to y) > 0$ we have $\|f(x_i) - f(y)\| \leqslant \varepsilon \|v\|$. Hence after averaging, for any $x \in X$ we have $\|Mf(x) - f(x)\| \leqslant \varepsilon \|v\|$ and so

$$\|Mf - f\|_\nu^2 \leqslant \varepsilon^2 \sum_i \nu(x_i) \|v\|^2 = \varepsilon^2 \|f\|_\nu^2$$

from which we deduce, using elementary Hilbertian geometry,

$$\langle Mf \mid f \rangle_\nu \geqslant (1-\varepsilon) \|f\|_\nu^2$$

hence $1 - \varepsilon \leqslant \sigma$, by definition of the spectrum of M.

So the representation π has no (S,ε)-invariant vectors for $\varepsilon < 1 - \sigma$, hence property (T) for G.

This ends the proof of the theorem. □

Random walks arising from generating sets. All of this quite simplifies when the random walk μ is a random walk on G arising from some finite symmetric set of generators.

Indeed, in this case, in the proof of the implication $(i) \Rightarrow (ii)$ above we do not need to consider lazy random walks, and we can take $k = 1$ so that $\lambda = \mu$ and $\alpha = 1/|S|$. Moreover, in this case the quotient random walk is trivial. So for any unitary representation π and any $f \in \mathcal{E}_\pi$ orthogonal to the constants we get

$$\langle Mf \mid f \rangle_\nu \leqslant \left(1 - \kappa^2/2\,|S|\right) \|f\|_\nu^2$$

where κ is a Kazhdan constant. So:

Proposition 14 – *Let G be a group with property (T), generated by a finite symmetric set S. Let κ be a Kazhdan constant for G with respect to S. Then the G-equivariant random walk on G arising from S has spectral gap at least $\kappa^2/2\,|S|$.*

In the reverse direction, it is clear from the proof of $(iii) \Rightarrow (i)$ that the spectral gap of the random walk on G arising from S is a Kazhdan constant for S.

If G' is a quotient of G and if S is a finite set generating G then we can also consider the random walk on G' arising from right multiplication by S. Then, using Proposition 7 we get

Proposition 15 – *Let G be a group with property (T), generated by a finite symmetric set S. Let κ be a Kazhdan constant for G with respect to S. Let G' be any quotient of G. Then the random walk on the Cayley graph of G' with respect to S has spectral gap at least $\kappa^2/2\,|S|$*

If G' is infinite, then 1 and -1 cannot belong to the spectrum of the ordinary random walk, and so the spectral gap provides a uniform control on the growth and cogrowth of all infinite quotients of a Kazhdan group (cf. [**?**]).

3 A spectral sufficient criterion for property (T)

Let us now state some sufficient criterion for property (T). This criterion enables us to transfer some spectral estimate on the bowls of radius 2 in a simplicial complex to the whole complex.

Let X be a simplicial complex and let x be a vertex of X. The *link* L_x of x in X in the graph whose vertices are the vertices of X linked to x by some edge; there is an edge between y and z in L_x if and only if xyz is a 2-face of X (if there are several such triangles we put as many edges).

By the spectral gap of a graph we mean the spectral gap of the random walk in which the transition probability between two vertices x and y is the number of edges joining x to y divided by the degree of x (cf. Example 4).

Theorem 16 – *Let X be a connected simplicial complex. Suppose that each vertex belongs to some 2-face. Suppose moreover that for each $x \in X$, the link L_x of x is connected and has a spectral gap greater than $1/2$. Then, any discrete group G acting freely on X with finite quotient has property (T).*

Thanks to this theorem, we get a criterion for property (T) of a group with a given presentation, which depends only on the relations of length 3 in the presentation. In particular, this criterion can be algorithmically checked.

Corollary 17 – *Let G be a group generated by a finite symmetric set S, with $e \notin S$. Let $L(S)$ be the graph with vertex set S and in which $\{s, s'\}$ is an edge if and only if $s^{-1}s' \in S$. Suppose that $L(S)$ is connected and has spectral gap greater than $1/2$. Then G has property (T).*

Proof of the corollary – Let $\langle S \mid R \rangle$ be a presentation of G. Add to R all relations of length 3 which hold in G (that is, which are consequences of R: this is still a presentation of G). Consider the Cayley complex X_0 of this presentation: the vertex set of X_0 is G; the edge set is $G.S$ i.e. for each $g \in G$, $s \in S$ there is an edge between g and gs; the face set is $G.R$ i.e. for each $g \in G$, for each $r = s_1 \ldots, s_k$ with $s_i \in S$, there is a face spanning the points $g, gs_1, gs_1s_2, \ldots, gs_1 \ldots s_k = g$. Define X by removing some 2-faces to this complex: keep only those 2-faces which express a relation of length 3 in the generators (and, if several 2-faces share the same boundary, keep only one). By definition, G acts freely on X with finite quotient. Now it is immediate to see that the graph $L(S)$ of the statement is isomorphic to the link at any point in X. (We have to exclude e from S since the link of a point does not contain this point.) $\square$

Remark 18 – It may not be easy, given a group presentation, to check whether or not $s^{-1}s' \in S$ for some $s, s' \in S$. However, since a quotient of a group with property (T) still has property (T), it is enough to apply the criterion above using only the relations of length 3 that are explicitly in some given presentation of the group. That is, if adding only the edges (s, s') for which one has some way to check that $s^{-1}s' \in S$ already gives a graph with spectral gap greater than $1/2$ then the group has property (T).

Note that it is easy to obtain presentations with lots of relations of length 3, since any presentation can be triangulated. Similarly, it is possible to ensure connectedness of $L(S)$ by replacing S by the set of words of length 1 and 2 on S. This criterion can be applied especially to random groups, showing that "generic" groups with "enough" relations have property (T) (cf. [Z]).

The value $1/2$ given in these statements cannot be improved. Indeed, consider the tiling of the Euclidean plane by equilateral triangles. The link of each point is a cycle of length 6, which has spectral gap 1/2 as can be checked. But the group of isometries of this tiling is isomorphic to $\mathbb{Z}^2$ for which property (T) fails.

Proof of Theorem 16 – The idea is the following: perform a random walk μ in X in which the transition probability from y to z is proportional to the number of triangles of X containing y and z. This random walk is such that two points y and z belong to the link of some point if and only if $\mu^{*2}(y \to z) > 0$; and (by definition of the links) they are joined by an edge in some link if and only if they belong to the same triangle i.e. $\mu(y \to z) > 0$. So if the spectral gaps of the links are greater than 1/2, then, by Proposition 9, the variance over the points joined by μ accounts for more than half the variance over the points joined by μ^{*2}. But this is a Poincaré inequality (Proposition 8), so the random walk has a spectral gap, hence property (T).

Namely, for $x \in X$ let $\nu(x)$ be two times the number of triangles (2-faces) containing x. For $x, y \in X, x \neq y$ let $\mu(x \to y)$ be $1/\nu(x)$ times the number of triangles containing both x and y. By construction, μ is a random walk on X, symmetric with respect to ν. It is G-invariant since G acts on the simplicial complex X sending triangles to triangles. Given a starting point, this random walk chooses some triangle containing this point and then jumps to one of the two other vertices of this triangle (hence the factor 2).

The graph of points joined by one step of the random walk is connected. Indeed, suppose to the contrary that there exist two points joined by an edge in X but not joined by a path of triangles; in this case each point is isolated in the link of the other one, which contradicts the connectedness assumption on the links (unless X is just two points with an edge, which is further excluded by the assumption that any point belongs to some triangle).

Now let $x \in X$ and consider points y, z in the link of x. We define an ordinary random walk μ_x on L_x, symmetric with respect to some measure ν_x, by setting $\nu_x(y)$ equal to the number of triangles containing x and y, and $\mu_x(y \to z)$ equal to $1/\nu_x(y)$ times the number of triangles containing x, y and z, for $y \neq z$. By definition of edges in the link, this is simply the natural random walk in the link.

Now suppose that each link L_x is connected, and let $\lambda(x)$ be the spectral gap of the natural random walk in L_x. By Proposition 9, since $\sum_y \nu_x(y) = \nu(x)$ we have

$$\frac{1}{\nu(x)} \sum_{(y,z) \in L_x \times L_x} \|f(y) - f(z)\|^2 \nu_x(y)\nu_x(z) \leqslant \frac{1}{\lambda(x)} \sum_{(y,z) \in L_x \times L_x} \|f(y) - f(z)\|^2 \nu_x(y)\mu_x(y \to z)$$

for any function f from L_x to any Hilbert space.

It is clear that $\nu_x(y) = \nu(x)\mu(x \to y)$ (where we extend ν_x and μ_x by 0 on points not in L_x): these are different ways of writing the same number of triangles. So for any fixed y and z we have

$$\sum_{x \in X} \frac{1}{\nu(x)} \nu_x(y)\nu_x(z) = \sum_{x \in X} \frac{1}{\nu(x)} \nu(y)\mu(y \to x)\, \nu(x)\mu(x \to z) = \nu(y)\mu^{*2}(y \to z)$$

which basically says that points y and z are connected by two steps of the random walk μ if they both belong to the link of some point; similarly,

$$\sum_{x \in X} \nu_x(y)\mu_x(y \to z) = \nu(y)\mu(y \to z)$$

since the number of triangles containing y and z is the number of triangles containing x, y and z for some x.

So if we know that for any x we have $\lambda(x) \geqslant \lambda$, we can sum the inequality above for all $(x, y, z) \in (X \times X \times X)/G$ and get

$$\sum_{(y,z)\in X\times X/G} \|f(y) - f(z)\|^2 \, \nu(y)\mu^{*2}(y \to z) \leqslant \frac{1}{\lambda} \sum_{(y,z)\in X\times X/G} \|f(y) - f(z)\|^2 \, \nu(y)\mu(y \to z)$$

for any G-equivariant function f into a Hilbert space with a action of G, or, in other words,

$$E_{\mu^{*2}}(f) \leqslant \frac{1}{\lambda} E_\mu(f)$$

which means that, if $\lambda > 1/2$, we can apply the criterion of Proposition 8. We deduce that the random walk has a spectral gap. Of course we conclude using Theorem 10. □

References

[Gh] É. Ghys, *Groupes aléatoires*, séminaire Bourbaki **916** (2003).

[V] A. Valette, *Nouvelles approches de la propriété* (T) *de Kazhdan*, séminaire Bourbaki **913** (2002).

[HV] P. de la Harpe, A. Valette, *La propriété* (T) *de Kazhdan pour les groupes localement compacts*, Astérisque **175**, SMF (1989).

[Gro] M. Gromov, *Random Walk in Random Groups*, Geom. Funct. Anal. **13** (2003), No. 1, 73–146, section 3.

[Z] A. Żuk, *Property (T) and Kazhdan constants for discrete groups*, Geom. Funct. Anal. **13** (2003), No. 3, 643–670.

[BS] W. Ballmann, J. Swiatkowski, *On L^2-cohomology and property* (T) *for automorphism groups of polyhedral cell complexes*, Geom. Funct. Anal. **7** (1997), 615–645.