Noncommutative geometry and particle physics, after Alain Connes

Edited by Yann Ollivier

The following is directly taken from the very well-written Chapter VI in Connes' 1994 book *Noncommutative geometry*; we focus on the relationship between noncommutative geometry and the standard model of particle physics. I am not an expert of either, so use with caution.

This is only an old part of Alain Connes' work on the subject, dating back to 1994.

Summary. [For simplicity I excluded the quarks and strong interaction.] Connes defines an algebraic extension of the notion of connection, exterior calculus, curvature tensor etc., from differential geometry. There is in particular a notion of connection for vector bundles over finite spaces. The Yang–Mills functional can be defined for such objects. When starting with a space $V \times F$ where V is a (Riemannian) manifold and $F = \{a, b\}$ is a two-point space, and considering a vector bundle isomorphic to $\mathbb{C}$ over a and $\mathbb{C} \otimes \mathbb{C}$ over b, connections over $V \times F$ have a U(1) part, a U(2) (instead of SU(2)) part and a discrete part given by two scalars (φ_1, φ_2). The Yang–Mills functional applied to this connection yields a Lagrangian very close to the one in the standard model (without quarks or strong interaction), with a few less free parameters: in particular, the action associated with the discrete part of the connection gives exactly the quartic Higgs potential together with its Yukawa coupling to the fermions, and correctly predicts the hypercharge of the Higgs field (and fixes its mass term). The "generations" of particles have a nice interpretation in this model.

[TODO: inclusion of quarks (involves quaternions), and passage from U(2) to SU(2) using the more recent papers by Connes.]

Reminder on the Yang–Mills action. Let ∂_μ be a reference connection for a vector bundle E on a Riemannian manifold M. Let A_μ be a linear form with values in GL(E) (e.g. for electromagnetism and spin-0 particles, E is the trivial 1-dimensional bundle, and A_μ is the vector potential). Consider the connection

$$\nabla_\mu = \partial_\mu + iA_\mu$$

then its curvature is

$$\nabla_{[\mu}\nabla_{\nu]} = \partial_{[\mu}\partial_{\nu]} + i\partial_{[\mu}A_{\nu]} - A_{[\mu}A_{\nu]} =: \partial_{[\mu}\partial_{\nu]} + iF_{\mu\nu}$$

where $\partial_{[\mu}\partial_{\nu]}$ is just the curvature of the reference connection. Note that for a 1-dimensional bundle E, the term $A_{[\mu}A_{\nu]}$ vanishes.

The Yang–Mills action is the following functional of A_μ:

$$\mathrm{YM}(A_\mu) := -\frac{1}{4}\,\mathrm{Tr}(F^*_{\mu\nu}F^{\mu\nu})$$

where the trace is taken in $\mathrm{GL}(E)$, and the metric is used to raise or lower the components μ and ν of tensors over the base manifold.

Thus, if the reference connection is flat, this is just the norm of the curvature form of the connection ∇, where the norm is defined using the metric g on the base and the Hilbert–Schmidt norm in the fibers.

1 Non-commutative Yang–Mills action

The goal is to define this Yang–Mills action in a more general setting, and to obtain something very close to the Lagrangian of the standard model, by using a noncommutative-geometric version of the above.

Basic data.
An $*$-algebra $\mathscr{A}$ with unit, acting on a Hilbert space $\mathscr{H}$. A selfadjoint unbounded operator D over $\mathscr{H}$, with compact resolvent. We assume that for any $f \in \mathscr{A}$, the commutator $[D,f]$ is bounded.

We assume that the eigenvalues of D grow like $\lambda_k \approx k^{1/d}$ for some number $d > 0$.

Basic example (Dirac operator).
$\mathscr{A}$ is the commutative algebra of complex-valued smooth functions on a compact spin manifold M, $\mathscr{H}$ is the Hilbert space of spinor fields over M, and $D = i\gamma^\mu\nabla_\mu = i\nabla\!\!\!/$ is the Dirac operator (with γ the gamma matrices defining a spin representation, and ∇ the spin connection). In this case for $f \in \mathscr{A}$ we have (exercise)

$$[D,f] = i\gamma^\mu\partial_\mu f = i\partial\!\!\!/ f.$$

where ∂f is the ordinary (de Rham) differential of f. Note that $\partial\!\!\!/ f$ is an operator on $\mathscr{H}$, but does not belong to $\mathscr{A}$ in general.

The Weyl estimate for the growth of the eigenvalues of the Laplacian $\Delta = D^2$ shows that the eigenvalues of D grow like $\lambda_k \approx k^{1/d}$ with $d = \dim M$.

So the operator $[D,f]$ acts on $\mathscr{H}$ as Clifford multiplication by $i\partial f$. Our next goal is to define, from $\mathscr{A}$ and D, an object which will play the role of the de Rham complex of differential forms, based on this identification between $[D,f]$ and ∂f.

We could use the universal anticommutative differential algebra over $\mathscr{A}$; however, we need an inner product on 1-forms and 2-forms that is related to the metric structure, and we want an action on $\mathscr{H}$ by identifying $\mathrm{d}f$ and $[D,f]$. We proceed as follows.

Exterior calculus over $\mathscr{A}$. Let $\Omega^*\mathscr{A}$ be the universal differential graded algebra over $\mathscr{A}$, i.e. the largest algebra containing $\mathscr{A}$ and all formal symbols $\mathrm{d}f$ for all $f \in \mathscr{A}$, with the relations $\mathrm{d}1 = 0$ and

$$\mathrm{d}(fg) = (\mathrm{d}f)g + f\mathrm{d}g$$

for all $f, g \in \mathscr{A}$. Explicitly, elements of $\Omega^*\mathscr{A}$ can be written as sums of terms $f_0\mathrm{d}f_1 \dots \mathrm{d}f_k$ using the relation above.

One can extend d into a derivation on $\Omega^*\mathscr{A}$ by imposing $\mathrm{d}^2 = 0$ and $\mathrm{d}(f\omega) = (\mathrm{d}f)\omega + f\mathrm{d}\omega$ for all $f \in \mathscr{A}$, $\omega \in \Omega^*\mathscr{A}$. This implies

$$\mathrm{d}(\omega\omega') = (\mathrm{d}\omega)\omega' + (-1)^{\deg\omega}\omega\mathrm{d}\omega'$$

for all $\omega, \omega' \in \Omega^*\mathscr{A}$. Let us also set

$$(\mathrm{d}f)^* := -\mathrm{d}(f^*)$$

(the reason being $\int f'g = -\int fg'$; and under the identification $\mathrm{d}f \leftrightarrow [D,f]$ below, we will have $[D,f]^* = -[D,f^*]$ indeed).

If $\mathscr{A}$ acts on a Hilbert space $\mathscr{H}$ as above, and D is a selfadjoint operator on $\mathscr{H}$, we would like to construct a differential algebra in which $\mathrm{d}f$ identifies with $[D,f]$ because the latter is Clifford multiplication by $i\partial f$. This can be done by a quotienting procedure as follows.

We can define a morphism of $\mathscr{A}$-$*$-algebras[1]

$$\pi : \Omega^*\mathscr{A} \to \mathrm{L}(\mathscr{H})$$

by sending $\mathrm{d}f$ to $[D,f]$. Unfortunately, $\pi(\omega) = 0$ does not in general imply $\pi(\mathrm{d}\omega) = 0$ so that the image does not have a *differential* algebra structure[2] (in particular $[D,\cdot]$ does not satisfy $[D,[D,\cdot]] = 0$ in general).

[1]This works because the defining relations $\mathrm{d}1 = 0$ and $\mathrm{d}(fg) = \dots$ of the universal differential $\mathscr{A}$-algebra are satisfied by $[D,\cdot]$.

[2]For instance, $\pi(\Omega^1\mathscr{A})$ contains the elements of the form $\sum f_i[D,g_i]$. One might be tempted to set $\mathrm{d}(f[D,g]) \overset{?}{:=} [D,f][D,g]$ but this is ill-defined. Indeed, in the basic example, we have $f\partial g + g\partial f = \partial(fg)$ so that we have $f[D,g] + g[D,f] = 1.[D,fg]$ in $\pi(\Omega^1\mathscr{A})$. Thus, the decomposition of an element as a sum $\sum f_i[D,g_i]$ is not unique; the definition above would yield $[D,f][D,g] + [D,g][D,f] = [D,1][D,fg] = 0$ but $[D,f][D,g] + [D,g][D,f]$ does not act on spinors as 0 (by Clifford calculus, it acts as pointwise multiplication by $-2g^{\mu\nu}\partial_\mu f \partial_\nu g$).

Let $J_0^k := \ker\pi \cap \Omega^k\mathscr{A}$, and let[3] $J_0 := \bigoplus J_0^k \subset \ker\pi$. It is a graded two-sided ideal of $\Omega^*\mathscr{A}$. Let $J = J_0 + \mathrm{d}J_0$, now J is stable by d and is (exercise) still a graded two-sided ideal of $\Omega^*\mathscr{A}$. Set

$$\Omega_D^* := \Omega^*\mathscr{A}/J$$

then we have

$$\Omega_D^k \simeq \pi(\Omega^k\mathscr{A})/\pi(\mathrm{d}(\ker\pi \cap \Omega^{k-1}\mathscr{A})).$$

By construction Ω_D^* inherits the differential graded algebra structure from $\Omega^*\mathscr{A}$. So the derivation d is still well-defined, and now coincides with $[D,\cdot]$, i.e.

$$\mathrm{d}(f_0[D,f_1][D,f_2]\dots[D,f_k]) = [D,f_0][D,f_1][D,f_2]\dots[D,f_k]$$

using the identification above of Ω_D^k as a quotient of $\pi(\Omega^k\mathscr{A}) \subset \mathrm{L}(\mathscr{H})$.

Assume that $\mathscr{A}$ acts faithfully on $\mathscr{H}$ (ie $\mathscr{A} \to \mathrm{L}(\mathscr{H})$ is injective). Then $\Omega_D^0 \simeq \mathscr{A}$ and $\Omega_D^1 \simeq \{\sum f_i[D,g_i]\} \subset \mathrm{L}(\mathscr{H})$. So in that case both Ω_D^0 and Ω_D^1 sit in $\mathrm{L}(\mathscr{H})$, and $\mathrm{d}f \in \Omega_D^1$ is exactly $[D,f]$. So in that

In $\pi(\Omega^k\mathscr{A}) \subset \mathrm{L}(\mathscr{H})$, we can define the inner product

$$\langle T_1, T_2\rangle_k := \mathrm{Tr}_\omega\left(T_1^* T_2 |D|^{-d}\right)$$

with Tr_ω the Dixmier trace[4]. After quotienting and completion, this defines an inner product on Ω_D^k, which we will reuse later.

Proposition 1.
In the basic example, the complex Ω_D^* is canonically isomorphic to the de Rham complex, and the scalar product above is, up to a factor $c(d)$, equal to $\langle\omega,\omega'\rangle = \int_M \omega\wedge *\omega'$.

[3]My edition of *Noncommutative geometry* states that $J_0 = \ker\pi$, but this is not true in general. For instance, in the basic example, $\ker\pi$ will contain elements of the form $\mathrm{d}f\mathrm{d}g + \mathrm{d}g\mathrm{d}f - c$ with $c = 2\langle\partial f,\partial g\rangle$ (the Clifford relation), which do not lie in J_0. Using $\ker\pi$ instead of J_0 results in a smaller object in the end, not equal to the de Rham complex in the basic example.

[4]By definition, the Dixmier trace $\mathrm{Tr}_\omega(T)$ of a compact operator T is $\lim_{N\to\infty}\frac{1}{\log N}\sum_{k\leqslant N}\mu_k$ with μ_k the eigenvalues of $|T| := (T^*T)^{1/2}$, arranged in decreasing order. Note that $\sum_{k\leqslant N}\mu_k = \sup\{\|Tp_E\|_1, \dim E = N\}$ with p_E the orthogonal projector onto E. The Dixmier trace is a trace, and, in particular, does not depend on the choice of the scalar product in $\mathscr{H}$.

Here the f_i and $[D,f_j]$ have been assumed to be bounded, and the eigenvalues of D have been assumed to grow like $k^{1/d}$, so that the limit exists.

In the basic example the Weyl asymptotic for the spectrum of $\Delta = D^2$ implies:

Proposition.
Let f be a smooth function on the compact d-dimensional spin manifold M. Then $\int_M f = c(d)\mathrm{Tr}_\omega(f|D|^{-d})$, with $c(d) = 2^{d-[d/2]}\pi^{d/2}\Gamma(d/2+1)$.

Let us prove, for instance, that $\mathrm{d}f\,\mathrm{d}g = -\mathrm{d}g\,\mathrm{d}f$ in Ω^2_D. As operators on $\mathscr{H}$, we have $\partial(fg) = f\partial g + g\partial f$, so that $\mathrm{d}(fg) - f\,\mathrm{d}g - g\,\mathrm{d}f \in \ker\pi \cap \Omega^1\mathscr{A}$. Now $\mathrm{d}(\mathrm{d}(fg) - f\,\mathrm{d}g - g\,\mathrm{d}f) = -\mathrm{d}f\,\mathrm{d}g - \mathrm{d}g\,\mathrm{d}f$ in $\Omega^*\mathscr{A}$, so by construction this will vanish in Ω^2_D.

Algebraic vector bundles and connections. The sections of a vector bundle over a manifold M form a module over the algebra $\mathscr{A}$ of smooth functions on M, and this module is finitely generated and projective[5] (Swan's theorem). Conversely, every such finitely generated projective $\mathscr{A}$-module has this form.

Moreover, given a Hermitian vector bundle over M, the pointwise scalar product of two sections of this bundle gives a function on M, i.e. an element of $\mathscr{A}$.

Definition 2 (Hermitian bundles).
Let E be a finitely generated projective right module over $\mathscr{A}$. A Hermitian structure *on E is a sesquilinear form $E \times E \to \mathscr{A}$ such that*[6]*:*

1. $\langle vf, wg\rangle = f^*\langle v,w\rangle g \qquad \forall f,g \in \mathscr{A},\, v,w \in E$
2. $\langle v,v\rangle \geqslant 0 \qquad \forall v \in E$
3. *E is self-dual for $\langle\cdot,\cdot\rangle$.*

Definition 3 (Connections).
A connection *on E is a $\mathbb{C}$-linear map $\nabla : E \to E \otimes_{\mathscr{A}} \Omega^1_D$ satisfying*

$$\nabla(vf) = (\nabla v)f + v \otimes \mathrm{d}f$$

for all $v \in E$ and $f \in \mathscr{A}$. The connection is said to be compatible *(with the Hermitian structure in E) if*

$$\mathrm{d}(\langle v,w\rangle) = -\langle \nabla v, w\rangle + \langle v, \nabla w\rangle$$

in Ω^1_D, for all $v,w \in E$.

(The minus sign comes from sesquilinearity and the fact that $(\mathrm{d}f)^* = -\mathrm{d}(f^*)$ in Ω^1_D.)

If ∇ is an ordinary Riemannian connection, then $i\nabla$ is a compatible connection in this sense.

One checks that one can extend the connection ∇ to higher degrees: let $E^*_D = E \otimes_{\mathscr{A}} \Omega^*_D$, and set

$$\nabla(v \otimes \omega) := (\nabla v)\omega + v \otimes \mathrm{d}\omega$$

[5] A projective module is a module E such that there exists E' such that $E \oplus E'$ is free. For instance, the tangent bundle of a manifold in $\mathbb{R}^n$ is projective because adding to it the normal bundle makes it free (turns it into a direct product bundle).

[6] For the second condition we would need to define positivity in $*$-algebras, which we omit as we never use it here.

for $v \in E$, $\omega \in \Omega_D^*$. Then, for $\xi \in E_D^k$, one has

$$\nabla(\xi\omega) = (\nabla\xi)\omega + (-1)^k \xi \mathrm{d}\omega$$

and then, the same computation as for Riemannian curvature yields that $\nabla^2 : E \to E \otimes_{\mathscr{A}} \Omega_D^2$ is "pointwise" i.e. $\mathscr{A}$-linear (actually Ω_D^*-linear):

$$\nabla^2(\xi f) = (\nabla^2 \xi) f$$

so that the operator ∇^2 can be seen as an element $\theta \in \mathrm{Hom}_{\mathscr{A}}(E, E \otimes_{\mathscr{A}} \Omega_D^2)$.

Now, E carries a Hermitian structure, which provides a Hilbert–Schmidt norm on $\mathrm{Hom}_{\mathscr{A}}(E,E)$ (which is finite since E is of finite rank); and we defined an inner product on Ω_D^2 above (using the Dixmier trace); together these define an inner product on $\mathrm{Hom}_{\mathscr{A}}(E, E \otimes_{\mathscr{A}} \Omega_D^2)$. Thus we can set

$$\mathrm{YM}(\nabla) := \|\theta\|^2_{\mathrm{Hom}_{\mathscr{A}}(E, E \otimes_{\mathscr{A}} \Omega_D^2)}$$

as desired. [So far I have not used the compatibility of the connection, which is used to prove gauge invariance, i.e. "parallel transport is an isometry".]

Note that if we replace D by λD for some $\lambda \in \mathbb{R}$, we have to replace ∇ with $\lambda\nabla$, and the Yang–Mills functional gets multiplied by λ^{4-d}.

Proposition 4.
In the basic example, with E a Hermitian vector bundle over M and ∇ a compatible connection on E, the Yang–Mills functional $\mathrm{YM}(\nabla)$ *is equal, up to a factor $c(d)$, to the integral over M of the square norm of the ordinary curvature tensor of* ∇.

Trivial one-dimensional bundle and electromagnetism. The trivial 1-dimensional bundle is given by $E = \mathscr{A}$ (in the basic example, $\mathbb{C}$-valued functions over the manifold). In this case, the map d from $E = \mathscr{A}$ to $E \otimes_A \Omega_D^1 = \Omega_D^1$ given by $f \mapsto \mathrm{d}f$ is a connection, and so is $\mathrm{d} + V$ for any fixed "vector potential" $V \in \Omega_D^1$. The connection $\mathrm{d} + V$ is compatible if and only if $V = V^*$. Its curvature is equal to $\mathrm{d}V + V^2 \in \Omega_D^2$. Beware that even if $\mathscr{A}$ is commutative, the term V^2 may not vanish in Ω_D^2—this requires both that $\mathscr{A}$ be commutative and that $f\mathrm{d}g = \mathrm{d}g\, f$ in Ω_D^1.

Let us rewrite the electromagnetic action in the current language. Here we have $E = \mathscr{A} = C^\infty(X_1, \mathbb{C})$ the trivial bundle of smooth functions over X_1. Let ∂ be the usual differential on functions on X_1. Let V be a real-valued 1-form on X_1 (electromagnetic vector potential) and consider the connection $\nabla_\mu := \partial_\mu + iV_\mu$. Then $i\nabla$ is indeed a connection in the sense of Definition 3, where we note that Ω_D^1 identifies to usual 1-forms by $\mathrm{d}f \in \Omega_D^1 \leftrightarrow i\partial f$. In this situation, we have $\mathrm{d}(fg) = f\mathrm{d}g + g\mathrm{d}f$ and this, together with commutativity, implies that the term V^2 vanishes.

Fermionic part of the action. Here we identify $\mathscr{A}$ with its image in $\mathrm{L}(\mathscr{H})$. The action of f on $\psi \in \mathscr{H}$ is denoted $f.\psi$.

Given the vector bundle E, fermions over E are sections of $E \otimes_{\mathscr{A}} \mathscr{H}$. Let endow $E \otimes_{\mathscr{A}} \mathscr{H}$ with the inner product

$$\langle v \otimes \psi, v' \otimes \psi' \rangle := \langle \psi, \langle v, v' \rangle . \psi' \rangle.$$

Given a connection ∇ over the $\mathscr{A}$-module E, one extends D into an operator $\not\nabla$ acting on $E \otimes_{\mathscr{A}} \mathscr{H}$ by

$$\not\nabla(v \otimes \psi) := \nabla v.\psi + v \otimes D\psi$$

for $v \in E$, $\psi \in \mathscr{H}$, where the first multiplication uses the facts that $\nabla v \in E \otimes_{\mathscr{A}} \Omega^1_D$ and that $\Omega^1_D \subset \mathrm{L}(\mathscr{H})$. Using that $\mathrm{d}f$ acts on $\mathscr{H}$ as $[D, f]$, we check that $\not\nabla(vf \otimes \psi) = \not\nabla(v \otimes f\psi)$ so this is well-defined.

Moreover, if ∇ is compatible, then $\not\nabla$ is selfadjoint.

We then define a Lagrangian for pairs (∇, Ψ) as

$$\mathscr{L}(\nabla, \Psi) := \lambda \, \mathrm{YM}(\nabla) + \langle \Psi, \not\nabla \Psi \rangle$$

for ∇ a compatible connection and $\Psi \in E \otimes_{\mathscr{A}} \mathscr{H}$. Here $\lambda \in \mathbb{R}$ is a coupling constant.

Note that the Yang–Mills term is the curvature of ∇ as a connection on the bundle E, not as an operator on spinors $E \otimes \mathscr{H}$.

2 Examples

Ultimately, we will consider a space $X = X_1 \times X_2$ where X_1 is a spin manifold, and X_2 is a finite set; $\mathscr{A}$ will simply be the set of functions of X. A connection will thus have a continuous part identical to a usual Yang–Mills connection, and a discrete part akin to a discrete derivative. As usual the continuous part of the connection will be the gauge fields; the discrete part of the connection will describe the Higgs field, which thus appears as a discrete gauge field.

We begin with the case of a two-point space alone.

2.1 Two-space point and quartic potential

Let $X = \{a, b\}$ be a two-point space, and let $\mathscr{A} = \mathbb{C} \oplus \mathbb{C}$ be the set of functions on X. Let $\mathscr{H} = \mathscr{H}_a \oplus \mathscr{H}_b$ be a direct sum of some Hilbert spaces over a and b, with the obvious action of $\mathscr{A}$ by scaling on each component.

Exterior calculus. For $f \in \mathscr{A}$, one would like to think of $\mathrm{d}f$ as a discrete derivative, so one would like to define D such that $\mathrm{d}f = [D, f]$ acts on $\mathscr{H}$ in a way related to $f(b) - f(a)$. This is possible by intertwining a and b by setting

$$D = \begin{pmatrix} 0 & M^* \\ M & 0 \end{pmatrix}$$

where $M : \mathscr{H}_a \to \mathscr{H}_b$ is some linear operator (e.g., the identity if $\mathscr{H}_a = \mathscr{H}_b$). Then we have

$$\mathrm{d}f = [D, f] = \begin{pmatrix} 0 & (f(b) - f(a))M^* \\ (f(a) - f(b))M & 0 \end{pmatrix}.$$

The algebra $\mathscr{A}$ is generated by 1 and 1_a with $1(a) = 1(b) = 1$, $1_a(a) = 1$, $1_a(b) = 0$. We have $\mathrm{d}1 = 0$ and $\mathrm{d}(1_a) = \left(\begin{smallmatrix} 0 & -M^* \\ M & 0 \end{smallmatrix}\right)$. Thus the set of 1-forms Ω^1_D is the set $g\mathrm{d}(1_a)$ for $g \in \mathscr{A}$, i.e., the set $\left(\begin{smallmatrix} 0 & -g(a)M^* \\ g(b)M & 0 \end{smallmatrix}\right)$.

The set of 2-forms Ω^2_D is generated by $\mathrm{d}1_a \mathrm{d}1_a = \left(\begin{smallmatrix} -M^*M & 0 \\ 0 & -MM^* \end{smallmatrix}\right)$ so that 2-forms are the matrices $g\mathrm{d}1_a\mathrm{d}1_a = \left(\begin{smallmatrix} -g(a)M^*M & 0 \\ 0 & -g(b)MM^* \end{smallmatrix}\right)$. Note that if $M \neq 0$ the map sending $\mathrm{d}f$ (as an element of the universal differential algebra $\Omega^*\mathscr{A}$) to $[D, f]$ is injective over Ω^1, so that there is no quotient involved in the definition of Ω^2_D; so Ω^2_D sits in $\mathrm{L}(\mathscr{H})$ and the norm on Ω^2_D is just the usual Hilbert–Schmidt norm[7].

Discrete connections. Let $E = E_a \oplus E_b$ be a vector bundle over $X = \{a, b\}$, with the obvious action of $\mathscr{A}$ by scaling on each component. Keep the space $\mathscr{H}$ and the operator $D = \left(\begin{smallmatrix} 0 & M^* \\ M & 0 \end{smallmatrix}\right)$ as above. Connections are $\mathbb{C}$-linear maps $E \to E \otimes_{\mathscr{A}} \Omega^1_D$, where we recall that Ω^1_D consists of all the operators $\left(\begin{smallmatrix} 0 & \lambda M^* \\ \mu M & 0 \end{smallmatrix}\right)$ with $\lambda, \mu \in \mathbb{C}$. Generically, the action of a connection ∇ on the section $(\xi_a, \xi_b) \in E$ can be written as

$$\nabla(\xi_a, \xi_b) = \begin{pmatrix} 0 & (\Phi_{ab}\xi_b - \xi_a) \otimes M^* \\ (\Phi_{ba}\xi_a - \xi_b) \otimes M & 0 \end{pmatrix}$$

where Φ_{ab}, Φ_{ba} are $\mathbb{C}$-linear maps from E_b to E_a and E_a to E_b, respectively. $\nabla(\xi_a, \xi_b)$ is an element of $E \otimes_{\mathscr{A}} \Omega^1_D$ with Ω^1_D a set of matrices on $\mathscr{H}$, so the matrix above is a 2×2 matrix sending vectors $h = \left(\begin{smallmatrix} h_a \\ h_b \end{smallmatrix}\right) \in \mathscr{H}$ to vectors in $E \otimes_{\mathscr{A}} \mathscr{H} = (E_a \otimes \mathscr{H}_a) \oplus (E_b \otimes \mathscr{H}_b)$; hence the coefficient in the first row, second column must send $\mathscr{H}_b$ to $(E_a \otimes \mathscr{H}_a)$, etc. The constraint for ∇ to be compatible yields $\Phi_{ab} = \Phi^*_{ba}$.

This represents the idea that connections are ways to take derivatives between infinitely close vector spaces E_a and E_b, with Φ_{ba} the Christoffel symbols in the direction from b to a.

[7] We use the usual trace instead of the Dixmier trace in finite dimension.

Setting $\zeta_a = \xi_a - \Phi_{ab}\xi_b$ and $\zeta_b = \Phi_{ba}\xi_a - \xi_b$, we see that $\nabla\xi = \zeta \otimes \mathrm{d}(1_a)$ where $\mathrm{d}(1_a) = \left(\begin{smallmatrix} 0 & -M^* \\ M & 0 \end{smallmatrix}\right)$ as before. Thus $\nabla\nabla\xi = (\nabla\zeta)\mathrm{d}(1_a)$. So we find the curvature of ∇ to be

$$\nabla^2 = \begin{pmatrix} (\Phi_{ab}\Phi_{ba} - \mathrm{Id}_{E_a}) \otimes M^*M & 0 \\ 0 & (\Phi_{ba}\Phi_{ab} - \mathrm{Id}_{E_b}) \otimes MM^* \end{pmatrix}.$$

Compare this to the intuition of curvature as parallel-transporting along a small loop (from a to b then back) and comparing to the original value.

Discrete Yang–Mills action. Let us compute the action

$$L(\nabla, \Psi) = \mathrm{YM}(\nabla) + \langle \Psi, \not\nabla \Psi \rangle$$

where Ψ is a (fermionic) section in $E \otimes_{\mathscr{A}} \mathscr{H}$.

Taking the (Hilbert–Schmidt) square norm of the curvature yields

$$\mathrm{YM}(\nabla) = \left(2\,\|\Phi\Phi^*\|_{\mathrm{HS}}^2 - 4\,\|\Phi\|_{\mathrm{HS}}^2 + n_a + n_b\right)\|M^*M\|_{\mathrm{HS}}^2$$

where $n_a = \dim E_a$, $n_b = \dim E_b$, and Φ is indifferently Φ_{ab} or Φ_{ba}. Note the quartic dependence on Φ.

Now for the fermionic part of the action. Fermions are sections of $E \otimes_{\mathscr{A}} \mathscr{H}$; it is enough to compute $\langle \Psi', \not\nabla\Psi \rangle$ for $\Psi = \xi_a \otimes \psi_a + \xi_b \otimes \psi_b$ with ξ a section of E and ψ a section of $\mathscr{H}$, and likewise for Ψ'. We have $\not\nabla\Psi = (\nabla\xi).\psi + \xi \otimes D\psi$ and since $D = \left(\begin{smallmatrix} 0 & M^* \\ M & 0 \end{smallmatrix}\right)$ we simply find

$$\not\nabla\Psi = \begin{pmatrix} \Phi^*\xi_b \otimes M^*\psi_b \\ \Phi\xi_a \otimes M\psi_a \end{pmatrix}$$

with $\Phi = \Phi_{ba}$, so that

$$\langle \Psi', \not\nabla\Psi \rangle = \langle \Phi\xi'_a, \xi_b \rangle \langle M\psi'_a, \psi_b \rangle + \langle \xi'_b, \Phi\xi_a \rangle \langle \psi'_b, M\psi_a \rangle.$$

Take for instance the case of the trivial vector bundle $E = \mathscr{A}$ (or, more exactly, $E = \mathbb{C}_a \oplus \mathbb{C}_b$ where $\mathbb{C}_a$, $\mathbb{C}_b$ are 1-dimensional Hermitian spaces isomorphic to $\mathbb{C}$ but with no preferred isomorphism). Then compatible connections are parametrized by a single complex number $\varphi \in \mathbb{C}$ (well-defined only up to phase) as

$$\nabla f = \begin{pmatrix} 0 & (\varphi^* f(b) - f(a))M^* \\ (\varphi f(a) - f(b))M & 0 \end{pmatrix}.$$

Applying the above we simply find

$$\mathrm{YM}(\nabla) = 2\left(|\varphi|^2 - 1\right)^2 \mathrm{Tr}((M^*M)^2)$$

which is a quartic potential. For the fermionic part we have $\mathbb{V} = \begin{pmatrix} 0 & \varphi^* M^* \\ \varphi M & 0 \end{pmatrix}$ so that we find

$$\langle \psi, \mathbb{V}\psi \rangle = 2\,\mathrm{Re}(\varphi \langle \psi_b, M\psi_a \rangle)$$

which is of Yukawa type.

As a second example, if $n_a = 2$ and $n_b = 1$, we have $\Phi_{ab} = \binom{\varphi_1}{\varphi_2}$ so that the connection is given by a pair ("doublet") of complex numbers. Then $\Phi_{ba}\Phi_{ab} = |\varphi_1|^2 + |\varphi_2|^2$ so that

$$\mathrm{YM}(\nabla) = \left(2\left(|\varphi_1|^2 + |\varphi_2|^2\right)^2 - 4\left(|\varphi_1|^2 + |\varphi_2|^2\right) + 3\right) \|M^* M\|^2_{\mathrm{HS}}.$$

In this case fermionic sections $\Psi \in E_a \otimes \mathscr{H}_a + E_b \otimes \mathscr{H}_b$ are represented by a pair ("doublet") (ψ_{a1}, ψ_{a2}) together with a "singlet" ψ_b, and

$$\langle \Psi, \mathbb{V}\Psi \rangle = 2\,\mathrm{Re}\left(\varphi_1 \langle M\psi_{a1}, \psi_b \rangle + \varphi_2 \langle M\psi_{a2}, \psi_b \rangle\right)$$

so that ψ_b is coupled to both ψ_{a1} and ψ_{a2}.

2.2 Product of a manifold and two points

Let V be a compact d-dimensional Riemannian spin manifold, and $F = \{a, b\}$ a two-point space as above. Let $\mathscr{H}_V$ be the space of spinors on V, with D_V the Dirac operator. Let $\mathscr{H}_F = \mathscr{H}_{Fa} \oplus \mathscr{H}_{Fb}$ as above, and let D_F be the operator acting on $\mathscr{H}_F$ by $\begin{pmatrix} 0 & M^* \\ M & 0 \end{pmatrix}$ as before. Let $\mathscr{A}_V$, $\mathscr{A}_F$ be the (commutative) algebra of complex functions on V and F.

Set $\mathscr{A} := \mathscr{A}_V \otimes \mathscr{A}_F$ and $\mathscr{H} := \mathscr{H}_V \otimes \mathscr{H}_F = \mathscr{H}_V \otimes \mathscr{H}_{Fa} \oplus \mathscr{H}_V \otimes \mathscr{H}_{Fb}$. Let

$$D := D_V \otimes \mathrm{Id} + \gamma_5 \otimes D_F$$

where γ_5 is the "fifth gamma matrix"[8] or "orientation $\mathbb{Z}/2$-grading" acting on $\mathscr{H}_V$.

Exterior calculus. Let $f = (f_a, f_b) \in \mathscr{A}$, where f_a and f_b are both functions over the manifold V. Then one checks that

$$\mathrm{d}f = [D, f] = \begin{pmatrix} i\partial f_a \otimes \mathrm{Id} & (f_b - f_a)\gamma_5 \otimes M^* \\ (f_a - f_b)\gamma_5 \otimes M & i\partial f_b \otimes \mathrm{Id} \end{pmatrix}$$

[8] The map γ from a vector space E to the algebra of operators on $\mathscr{H}$ extends to $\bigwedge^d E$; the matrix γ_5 is the image of the positively-oriented unit element of $\wedge^d E$ with E the cotangent space to the spin manifold. If $e_1, \ldots, e_d$ is any positively-oriented orthonormal basis of E, then γ_5 is equal to $\gamma(e_1) \cdots \gamma(e_d)$.

The main point of γ_5 is that it anticommutes with D_V, so that $D^2 = D_V^2 \otimes \mathrm{Id} + \mathrm{Id} \otimes D_F^2$ as is expected from the behavior of Dirac operators under direct products. It moreover bears an analogy with the direct product of differential algebras, where one must define $\mathrm{d}(\omega_1 \otimes \omega_2) = \mathrm{d}\omega_1 \otimes \omega_2 + (-1)^{\deg \omega_1} \omega_1 \otimes \omega_2$.

One could of course start with $D_V \otimes \mathrm{Id} + \mathrm{Id} \otimes D_F$ and apply the general construction, but this would imply drastic quotienting in the definition of the space Ω_D.

acting on $\mathscr{H}_1 \otimes \mathscr{H}_{2a} \oplus \mathscr{H}_1 \otimes \mathscr{H}_{2b}$. Consequently, the set of 1-forms $\Omega^1_D = \{\sum g_i \mathrm{d} f_i\}$ is the set of all operators of the form

$$\begin{pmatrix} i\omega_a \otimes \mathrm{Id} & \delta_a \gamma_5 \otimes M^* \\ \delta_b \gamma_5 \otimes M & i\omega_b \otimes \mathrm{Id} \end{pmatrix}$$

where δ_a and δ_b are arbitrary functions on the manifold V, and ω_a and ω_b are (the Clifford image of) arbitrary 1-forms on V.

Let us now turn to Ω^2_D. A priori, elements of Ω^2_D can be written as

$$\begin{pmatrix} -\alpha_a \otimes \mathrm{Id} + h_a \otimes M^* M & i\gamma_5 \beta_a \otimes M^* \\ i\gamma_5 \beta_b \otimes M & -\alpha_b \otimes \mathrm{Id} + h_b \otimes M M^* \end{pmatrix}$$

where h_a and h_b are functions on V, β_a and β_b are (the Clifford image of) 1-forms on V, and α_a and α_b are degree-2 elements of the Clifford bundle over V. However, the definition of Ω^2_D involves a quotient by by $\mathrm{d}(\ker \Omega^1 \mathscr{A} \to \Omega^1_D)$. In the case of a manifold, we have seen that this quotient produces the de Rham complex. This means that α_a and α_b are ordinary 2-forms on V—i.e., degree-2 elements of the Clifford bundle quotiented by functions on V. In particular, if M^*M is a multiple of the identity, the functions h_a and h_b above will be absorbed by the quotient as well.

So from now on we assume $MM^* \neq \lambda \mathrm{Id}$. Let $(MM^*)_0$ denote the projection of MM^* onto the orthogonal of $\{\lambda \mathrm{Id}\}$ (i.e. the traceless part of MM^*). Thus elements of Ω^2_D are described as

$$\begin{pmatrix} -\alpha_a \otimes \mathrm{Id} + h_a \otimes (M^* M)_0 & i\gamma_5 \beta_a \otimes M^* \\ i\gamma_5 \beta_b \otimes M & -\alpha_b \otimes \mathrm{Id} + h_b \otimes (M M^*)_0 \end{pmatrix}$$

where α_a, α_b are usual 2-forms on V, β_a, β_b are 1-forms and h_a, h_b are functions.

In this computation, we have implicitly used the commutation relation of γ_5 with ∇f. If we had used something else instead of γ_5 the space Ω^2_D would be less intuitive.

Indeed, differentiating $\mathrm{d}(fg) = (\mathrm{d}f)g + f\mathrm{d}g$ we find $0 = \mathrm{d}((\mathrm{d}f)g) + \mathrm{d}f\mathrm{d}g$ in Ω^2_D; computing explicitly, we find $\mathrm{d}((\mathrm{d}f)g) + \mathrm{d}f\mathrm{d}g = \begin{pmatrix} (-\partial_a f \partial_a g - \partial_a g \partial_a f) \otimes \mathrm{Id} & 0 \\ 0 & (-\partial_b f \partial_b g - \partial_b g \partial_b f) \otimes \mathrm{Id} \end{pmatrix}$ so that we put no more relations in Ω_D than the de Rham complex, but the off-diagonal terms would not vanish if we used 1 instead of γ_5.

Proposition 5.
The square norm of an element in Ω^2_D as above is given by $1/c(d)$ times

$$\int_V \left(n_a \|\alpha_a\|^2 + n_b \|\alpha_b\|^2\right) + \mathrm{tr}(M^*M) \int_V \left(\|\beta_a\|^2 + \|\beta_b\|^2\right) + \mathrm{tr}\left((M^*M)_0^2\right) \int_V \left(\|h_a\|^2 + \|h_b\|^2\right)$$

with $n_a = \dim \mathscr{H}_{Fa}$, $n_b = \dim \mathscr{H}_{Fb}$ and $(M^*M)_0$ the trace-free part of M^*M.

Connections and curvature. A vector bundle E on $X = V \times \{a,b\}$ is given by two ordinary vector bundles E_a and E_b on V. A connection on X is given by two connections ∇_a and ∇_b on E_a and E_b together with Φ_{ab} and Φ_{ba} as before for each point of V, so that Φ_{ab} is a section of $E_b^* \otimes_V E_a$ and likewise for Φ_{ba}.

Explicitly, if $\xi = (\xi_a, \xi_b)$ is a section of E, its derivative $\nabla\xi$ is given by

$$\nabla\xi = \begin{pmatrix} i\nabla_a\xi_a \otimes \mathrm{Id} & \gamma_5(\Phi_{ab}\xi_b - \xi_a)\otimes M^* \\ \gamma_5(\Phi_{ba}\xi_a - \xi_b)\otimes M & i\nabla_b\xi_b \end{pmatrix}.$$

To compute the curvature of such a connection, one computes $\nabla^2\xi$ by decopmosing $\nabla\xi$ into elements of the form $\zeta \otimes \mathrm{d}f$ with ζ a section of E. In the end the curvature is

$$\nabla^2 = \begin{pmatrix} -\nabla_a^2 \otimes \mathrm{Id}_{\mathscr{H}_{2a}} + (\Phi_{ab}\Phi_{ba} - \mathrm{Id}_{E_a}) \otimes (M^*M)_0 & -i\gamma_5 \nabla\Phi_{ab} \otimes M^* \\ -i\gamma_5\nabla\Phi_{ba}\otimes M & -\nabla_b^2 \otimes \mathrm{Id}_{\mathscr{H}_{2b}} + (\Phi_{ba}\Phi_{ab} - \mathrm{Id}_{E_b}) \otimes (MM^*)_0 \end{pmatrix}$$

where the connection acts on Φ_{ab} by the usual extension of connections to tensors, i.e., $\nabla\Phi_{ab} = (\nabla_b^* \otimes 1 + 1 \otimes \nabla_a)\Phi_{ab}$ seeing Φ_{ab} as a section of $E_b^* \otimes_V E_a$ over the manifold V, and likewise for Φ_{ba}.

From this and Proposition 5 the computation of the action is straightforward. The terms obtained are:

- n_a and n_b times, respectively, the usual Yang–Mills terms for ∇_a and ∇_b.
- The quartic term

$$\left(2\left\|\Phi\Phi^*\right\|_{\mathrm{HS}}^2 - 4\left\|\Phi\right\|_{\mathrm{HS}}^2 + \dim E_a + \dim E_b\right)\left\|(M^*M)_0\right\|_{\mathrm{HS}}^2.$$

- A kinetic Φ term

$$2g^{\mu\nu}\,\mathrm{Tr}(\nabla_\mu\Phi_{ab}\nabla_\nu\Phi_{ba})\left\|M\right\|_{\mathrm{HS}}^2$$

- The fermionic term is obtained from $\mathbb{V} = \begin{pmatrix} i\mathbb{V}_a \otimes \mathrm{Id}_{\mathscr{H}_{Fa}} & \gamma_5\Phi_{ab}\otimes M^* \\ \gamma_5\Phi_{ba}\otimes M & i\mathbb{V}_b\otimes \mathrm{Id}_{\mathscr{H}_{Fb}} \end{pmatrix}$ acting on $E \otimes_{\mathscr{A}} \mathscr{H}_V \otimes_{\mathscr{A}} \mathscr{H}_F$, where in the usual way $\mathbb{V}_a = \nabla_{a\mu} \otimes \gamma^\mu + \mathrm{Id} \otimes \gamma^\mu \partial_{a\mu}$ acts on the E_a-valued spinors $E_a \otimes \mathscr{H}_V$ over V, and likewise for $\mathbb{V}_b$. One gets the usual kinetic terms for E-valued spinors, plus Yukawa-type terms coupling the a and b components using $\Phi_{ab} \otimes M$.

Standard model without quarks. Let us take $E_{Fa} = \mathbb{C}$ and $E_{Fb} = \mathbb{C}+\mathbb{C}$. Let us also take $\dim \mathscr{H}_{Fa} = \dim \mathscr{H}_{Fb} = N$, the number of generations in the standard model. Remember that $\mathscr{H}_V$ is just the space of spinors on V. Now fermions $\Psi \in E \otimes_{\mathscr{A}} \mathscr{H}_V \otimes_{\mathscr{A}} \mathscr{H}_F$ are given by $(\psi_a, (\psi_{b1}, \psi_{b2}))$ where each ψ is made of N spinors on V. We leave the operator M unspecified. The N

components of ψ_a are identified with the right-handed leptons (electron, muon, tau). The N components of ψ_{b1} are the left-handed neutrinos, and the N components of ψ_{b2} are the left-handed electron, muon, tau.

The fiber above a is 1-dimensional; thus, the connection ∇_a can be written as $\nabla_a = \partial_a + i\beta$ where β is a one-form on V (vector potential). This connection is compatible (with the inner product in $\mathbb{C}$, i.e., with the U(1) gauge group) if and only if B is real.

The fiber above b is 2-dimensional (fiber $\mathbb{C}^2$). Thus we can write $\nabla_b = \partial_b + i\omega$ where ω is a self-adjoint 2×2 matrix of 1-forms on V (the Christoffel symbols of ∇_b).

The discrete part of the connection is given by a vector $\Phi_{ba} = (\varphi_1, \varphi_2)$ representing the ways to send $\mathbb{C}$ to $\mathbb{C}+\mathbb{C}$, with $\Phi_{ab} = \Phi_{ba}^*$. The action of the connection ∇ on Φ_{ba} is by $\nabla = \partial + i\omega - i\beta$.

If one sets

$$B := -\beta \qquad W := 2\omega - \beta$$

one finds that the various couplings reproduce those observed in the standard model *with the correct values of the hypercharges*[9].

Note that since the image of Φ_{ba} in $\mathbb{C}^2$ is one-dimensional, the part of ψ_b which is orthogonal to Φ_{ba} does not interact with ψ_a. By definition this part of ψ_b is the (massless) neutrino, while the part of ψ_b collinear with Φ_{ba} is the massive lepton. The matrix M represents the interactions between generations; by diagonalizing M one gets three eigenvectors which are the electron, muon and tau.

Note that since the quartic term (giving the Higgs mass) depends on MM^* while the Yukawa term (giving the lepton masses if the Higgs field is fixed) depends on M, there is a relationship between the mass of the Higgs field and that of the leptons; namely, if one rescales φ so as to fix the degree-2 term in φ and then rescales the lepton basis so as to fix the lepton masses, then the degree-4 term in φ^4 is fixed.

The way we have defined compatible connections, the gauge group is $\mathrm{U}(1)\times\mathrm{U}(2)$ (not $\mathrm{U}(1)\times\mathrm{SU}(2)$). This means that there are too many degrees of freedom for ω compared to what is known. One has to impose the additional constraint $\mathrm{Tr}(\omega) = \beta$. [This seems to have been fixed in more recent versions of the model.]

All this is Euclidean, not Lorentzian—indeed we have used an L^2 norm on spinors, which is not Lorentz-invariant.

Quarks. To come.

[9] Identification of β and ω with the usual fields is only up to scaling; so actually it is the relationship between the hypercharges of left-handed leptons, right-handed leptons and Higgs field which is recovered, i.e. only the hypercharge of the Higgs.