# Some small cancellation properties of random groups

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**Abstract**

We work in the density model of random groups. We prove that they satisfy an isoperimetric inequality with sharp constant $1 - 2d$ depending upon the density parameter $d$. This implies in particular a property generalizing the ordinary $C'$ small cancellation condition, which could be termed "macroscopic cancellation". This also sharpens the evaluation of the hyperbolicity constant $\delta$.

As a consequence we get that the standard presentation of a random group at density $d < 1/5$ satisfies the Dehn algorithm and Greendlinger's Lemma, and that it does not for $d > 1/5$.

For this we establish a version of the local-global principle for hyperbolic spaces (Cartan-Hadamard-Gromov theorem) involving arbitrarily small loss in the isoperimetric constant.

## Statements

Gromov introduced in [Gro93] the so-called *density model* of random groups, which allows the study of generic groups with a very precise control on the number of relators put in the group, depending on a density parameter $d$.

A set of $m$ generators $a_1, \ldots, a_m$ being fixed, this model consists in choosing a (large) length $\ell$ and a density parameter $0 \leqslant d \leqslant 1$, and choosing at random a set $R$ of $(2m-1)^{d\ell}$ reduced words of length $\ell$. The random group is then the group given by the presentation $\langle a_1, \ldots, a_m \mid R \rangle$. (Recall a word is *reduced* if it does not contain a generator immediately followed by its inverse).

In this model, we say that a property occurs *with overwhelming probability* if its probability of occurrence tends to 1 as $\ell \to \infty$ (everything else being fixed).

The basic intuition behind the model is that at density $d$, subwords of length $(d - \varepsilon)\ell$ of the relators will exhaust all possible reduced words of this length. Also, at density $d$, with overwhelming probability there are two relators sharing a subword of length $(2d - \varepsilon)\ell$. We refer to [Gro93], [Ghy03] or [Oll-c] for a general discussion on random groups and the density model.

The interest of this way to measure the number of relators in a presentation is largely established by the following foundational theorem, due to Gromov ([Gro93], see also [Oll04]).

**THEOREM 1 (M. GROMOV)** –
*If $d < 1/2$, with overwhelming probability a random group at density $d$ is infinite and hyperbolic.*
*If $d > 1/2$, with overwhelming probability a random group at density $d$ is either $\{e\}$ or $\mathbb{Z}/2\mathbb{Z}$.*

(Occurrence of $\mathbb{Z}/2\mathbb{Z}$ of course corresponds to even $\ell$.)

Other properties of random groups are known, some of which depending on the density parameter (works of Arzhantseva, Champetier, Gromov, Ollivier, Ol'shanskiĭ, Żuk; see references in [Ghy03, Oll04, Oll-c]). The construction can be modified and iterated in various ways to achieve specific goals [Gro03].

Hyperbolicity for $d < 1/2$ is achieved by proving that van Kampen diagrams satisfy some isoperimetric inequality (we refer to [LS77] for definitions about van Kampen diagrams and to [Sho91] for the equivalence between hyperbolicity and isoperimetry of van Kampen diagrams). The main result of this paper is a sharp version of this isoperimetric inequality.

**THEOREM 2** – *For every $\varepsilon > 0$, with overwhelming probability, every reduced van Kampen diagram $D$ in a random group at density $d$ satisfies*

$$|\partial D| \geqslant (1 - 2d - \varepsilon)\,\ell\,|D|$$

This was already known to hold for diagrams of size bounded a priori (see Theorem 14), but the passage to all diagrams involves the local-global hyperbolic principle of Gromov (see e.g. [Pap96] for a constructive statement), which implies a substantial loss in the constants. After using this, the only constant available for all diagrams was something like $(1 - 2d)/10^{20}$. We solve the problem by giving a variant of the principle best suited to our needs (Theorem 8), which may have independent interest.

This inequality is sharp: indeed, at density $d$ there are very probably two relators sharing a subword of length $(2d - \varepsilon)\ell$, so that they can be arranged to form a 2-face van Kampen diagram of boundary length $2(1-2d+\varepsilon)\ell$. At density $d$ one can always glue some new relator to any diagram along a path of length $(d-\varepsilon)\ell$, so that adding relators to this example provides an arbitrarily large diagram with the same isoperimetric constant.

**COROLLARY 3** – *At density $d$, with overwhelming probability the hyperbolicity constant of a random group satisfies $\delta \leqslant 4\ell/(1 - 2d)$.*

**COROLLARY 4** – *For every $\varepsilon > 0$, with overwhelming probability, random groups at density $d$ satisfy the following: Let $D_1$ and $D_2$ be two reduced van Kampen* *diagrams, both of them homeomorphic to a disk. Suppose that their boundaries share a common reduced subword $w$. Suppose moreover that the diagram $D = D_1 \cup_w D_2$ obtained by gluing $D_1$ and $D_2$ along $w$ is still reduced. Then we have*

$$|w| \leqslant d\left(|\partial D_1| + |\partial D_2|\right)(1+\varepsilon)$$

When $D_1$ and $D_2$ each consist of only one face, this exactly states that random groups satisfy the $C'(2d)$ small cancellation property (which implies hyperbolicity only when $d < 1/12$). So this property is a kind of "macroscopic cancellation" (though not "small" cancellation when $d$ is close to $1/2$).

Our last application of Theorem 2 has to do with the Dehn algorithm and Greendlinger's Lemma, which are classical properties considered in combinatorial group theory (see [LS77], [Gre60]).

There are several versions of Greendlinger's Lemma. We will not use the strongest version which holds for $C'(1/6)$ presentations ([LS77], Theorem V.4.5). The exact property we will use is the following.

Given a face $f$ of a van Kampen diagram $D$, a *countour segment* of $f$ in $D$ is a subset of edges of $\partial f \cap \partial D$ which are consecutive in the boundary path of $D$.

**Definition 5 (Greendlinger's property)** – *We say that a group presentation satisfies the* Greendlinger property *if the following holds: For any reduced van Kampen diagram $D$ w.r.t. the presentation, with reduced boundary word, either $D$ has only one face or there exist at least two faces of $D$ having contour segments of lengths more than half their respective lengths.*

Of course this implies that Dehn's algorithm works.

One might expect from Theorem 2 that the Dehn algorithm holds as soon as $d < 1/4$. Indeed, $d < 1/4$ implies that some face of any reduced diagram has at least $\ell/2$ boundary edges; but these might not be consecutive. Actually the critical density is $1/5$.

**Theorem 6** – *If $d < 1/5$, with overwhelming probability, the standard presentation of a random group satisfies the Dehn algorithm and the Greendlinger property.*

*More precisely, for any $\varepsilon > 0$, with overwhelming probability, in every reduced van Kampen diagram with reduced boundary word, with at least two faces, there are at least two faces having a contour segment of length more than $\frac{\ell}{2} + \frac{\ell}{2}(1 - 5d - \varepsilon)$.*

*If $d > 1/5$, with overwhelming probability, the standard presentation of a random group does not satisfy the Dehn algorithm nor the Greendlinger property.*

See p. 13 for a simple example of a van Kampen diagram violating the Greendlinger property when $d > 1/5$.

**Discussion of the results.** The interest of the sharp constant depending on density in Theorem 2, compared to the $10^{20}$ times smaller previous estimate, is not only aethestic. Let us stress that the Dehn algorithm could not be obtained with the previous constant, if only for the reason that $(1-2d)/10^{20}$ is never greater than 1/2... So the improvement allows qualitative progress.

Both Theorem 2 and the Greendlinger property will be crucially used in [OW] to show that random groups at densities $< 1/6$ act freely cocompactly on $CAT(0)$ cube complexes and satisfy the Haagerup property.

Corollary 4 is probably unimportant but might justify to some extent the term "cancellation on average" applied to the density model (although this is certainly not "small cancellation on average", since when $d$ is close to 1/2 the cancellation becomes arbitrarily large).

The estimate of the hyperbolicity constant in Corollary 3 is of course not qualitatively different from the previous, $10^{20}$ times larger one.

Theorem 6 refers to the random presentation obtained by applying directly the definition of the density model. Note that in any $\delta$-hyperbolic group, the set of words of length at most $8\delta$ representing the identity constitutes a presentation of the group satisfying the Dehn algorithm ([Sho91], Theorem 2.12); however, this set of words is quite large, and computing it is feasible but tedious. Moreover this set of words does not in general satisfy the Greendlinger property, which is what is really needed in lots of applications.

What happens at $d = 1/5$ is not known (just as what happens for infiniteness or triviality at $d = 1/2$), but probably depends on more precise subexponential terms in the number of relators of the presentation, and so might not be very interesting.

Theorem 2 seems to remain valid in more general random group models when the lengths of the relators are not the same but lie within some bounded ratio (see [Oll04]). However I do not know if this is the case for Theorem 6.

Theorem 2 may also help show that random groups at different densities are indeed different.

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# Local-global principles

Following Gromov ([Gro87], 2.3.F, 6.8.M), there have been lots of somewhat different phrasings of the local-global principle for hyperbolic groups (chapter 8 of [Bow91], [Ols91], [Bow95], [Pap96]). This principle states that to ensure hyperbolicity, it is enough to check the isoperimetric inequality on a *finite* number of diagrams.

We give here a version which can be very neatly applied in our context, and which involves arbitrarily small loss in the isoperimetric constant. Though this version is not difficult to prove using previously stated results, it does not seem to be a formal corollary thereof.

**Definition 7** – *Let $D$ be a van Kampen diagram with respect to some presentation. The* area *$\mathcal{A}(D)$ of $D$ is the sum of the boundary lengths of all faces of $D$.*

We have advocated elsewhere ([Oll-a], [Oll-b]) that this is the right way to measure area in a context of linear isoperimetric inequalities involving relators with very different lengths. That it allows a formulation of the local-global principle without loss in the constant is a further argument in this direction.

**Theorem 8** – *Let $G = \langle a_1, \ldots, a_m \mid R \rangle$ be a finite group presentation and let $\ell_1, \ell_2$ be the minimal and maximal lengths of a relator in $R$.*

*Let $P$ be a class of van Kampen diagrams, such that any subdiagram of a diagram in $P$ lies in $P$.*

*Let $C > 0$. Choose $\varepsilon > 0$. Suppose that for some $K \geqslant 10^{50} (\ell_2/\ell_1)^3 \varepsilon^{-2} C^{-3}$, any van Kampen diagram $D$ in $P$ of area at most $K\ell_2$ satisfies*

$$|\partial D| \geqslant C\,\mathcal{A}(D)$$

*Then any van Kampen diagram $D$ in $P$ satisfies*

$$|\partial D| \geqslant (C - \varepsilon)\,\mathcal{A}(D)$$

*In particular, if $P$ is such that for each reduced word $w$ representing the identity in $G$, there is at least one diagram in $P$ spanning $w$, then $G$ is hyperbolic.*

It is not clear whether $\ell_2/\ell_1$ really has an impact on the constants.

Typical useful examples of the class $P$ are “reduced”, or “of minimal area”, or “of minimal number of faces” (minimal for a given boundary word).

This theorem may allow to extend the scope of the density model by taking relators of length between $\ell$ and $\ell^{1+\alpha}$ for some positive $\alpha$, instead of taking relators of length exactly $\ell$ (see the discussions in [Oll04]).

We are going to state closer and closer propositions to the theorem. The first one is a variant on Papasoglu’s exposition [Pap96] as modified in [Oll04].

Let $X$ be a complex of dimension 2. A *circle drawn in $X$* is a sequence of consecutive edges such that the endpoint of the last edge is the starting point of the first one. A *disk drawn in $X$* is a cellular map (maybe dimension-decreasing) from a cellular disk to $X$.

Let $f$ be a face of $X$. The *combinatorial length* $L_c$ of $f$ is defined as the number of edges of its boundary. The *combinatorial area* $A_c$ of $f$ is defined as $L_c(f)^2$.

Let $D$ be a disk drawn in $X$. The *combinatorial length* $L_c$ of $D$ is the length of its boundary. The *combinatorial area* $A_c$ of $D$ is the sum of the combinatorial areas of its faces.

We then have ([Oll04], Proposition 42, p. 666):

**Proposition 9** – *Let $X$ be a complex of dimension 2, simply connected. Suppose that a face of $X$ has at most $\ell$ edges. Let $P$ be a property of disks in $X$ such that any subdisk of a disk having $P$ also has $P$.*

*Suppose that for some integer $k \geqslant 10^{10}\ell$, any disk $D$ drawn in $X$ having $P$, whose combinatorial area $A_c(D)$ lies between $k^2/4$ and $480k^2$ satisfies*

$$L_c(D)^2 \geqslant 2 \cdot 10^{14}\, A_c(D)$$

*Then any disk $D$ drawn in $X$, having $P$, with $A_c(D) \geqslant k^2$, satisfies*

$$L_c(D) \geqslant A_c(D)/10^4 k$$

This allows to prove one more step:

**Proposition 10** – *Let $G = \langle a_1, \ldots, a_m \mid R \rangle$ be a finite presentation and let $\ell_1, \ell_2$ be the minimal and maximal lengths of a relator in $R$.*

*Let $P$ be a class of van Kampen diagrams, such that any subdiagram of a diagram in $P$ lies in $P$.*

*Let $C > 0$. Suppose that for some $K \geqslant 10^{23}\,(\ell_2/\ell_1)\, C^{-2}$, any van Kampen diagram $D$ in $P$ of area $\mathcal{A}(D)$ at most $K\ell_2$ satisfies*

$$|\partial D| \geqslant C\, \mathcal{A}(D)$$

*Then any van Kampen diagram $D$ in $P$ satisfies*

$$|\partial D| \geqslant C'\, \mathcal{A}(D)$$

*with $C' = C\,(\ell_1/\ell_2)/10^{15}$.*

**Proof** – We have $A_c(D)/\ell_2 \leqslant \mathcal{A}(D) \leqslant A_c(D)/\ell_1$ for any diagram $D$ in class $P$ (remember $\mathcal{A}(D)$ is the sum of the lengths of the faces whereas $A_c(D)$ is the sum of the squares of these lengths).

Set $k^2 = K\ell_1\ell_2/480$. Let $D$ be a van Kampen diagram such that $k^2/4 \leqslant A_c(D) \leqslant 480k^2$. We have $\mathcal{A}(D) \leqslant A_c(D)/\ell_1 \leqslant K\ell_2$. So the assumption of the proposition states that $L_c(D) = |\partial D| \geqslant C\mathcal{A}(D)$. Thus

$$L_c(D)^2 \geqslant C^2\mathcal{A}(D)^2 \geqslant C^2 A_c(D)^2/\ell_2^2 \geqslant C^2 A_c(D)k^2/4\ell_2^2 = A_c(D)C^2K(\ell_1/\ell_2)/1920$$

So if $k \geqslant 10^{10}\ell_2$ and $C^2K(\ell_1/\ell_2)/1920 \geqslant 2 \cdot 10^{14}$ then the assumptions of Proposition 9 are fulfilled. Taking $K = 10^{23}\,(\ell_2/\ell_1)/C^2$ is enough to ensure this is the case. The consequence of Proposition 9 is then that

$$|\partial D| = L_c(D) \geqslant A_c(D)/10^4 k \geqslant \mathcal{A}(D)\, \ell_1/10^4 k$$

and unwinding the constants shows that $\ell_1/10^4 k \geqslant C\,(\ell_1/\ell_2)/10^{15}$. □

Going on with our approximations of Theorem 8, we now know that there exists an isoperimetric constant $C'$, but its value may be much smaller than the original constant $C$. We solve the problem by a kind of bootstrapping: we will re-do some kind of local-global passage, using our knowledge of hyperbolicity of the group. This will allow to keep the constants tight.

We need a lemma from [Oll-a].

The *distance to boundary* of a face of a van Kampen diagram is the minimal length of a sequence of faces adjacent by an edge, beginning with the given face and ending with a face adjacent to the boundary (so that a boundary face is at distance 1 from the boundary).

Let $C'$ be the isoperimetric constant provided by Proposition 10, so that any diagram $D$ in $P$ satisfies $|\partial D| \geqslant C'\mathcal{A}(D)$. Set

$$\alpha = 1/\log(1/(1-C')) \leqslant 1/C'$$

The following is Lemma 10 of [Oll-a], where we replaced "minimal" by "in class $P$".

**Lemma 11** – *Let $D$ be a van Kampen diagram in class $P$. Then $D$ can be partitioned into two diagrams $D'$, $D''$ by cutting it along a path of length at most $\ell_2 + 2\alpha\ell_2 \log(\mathcal{A}(D)/\ell_2)$ with endpoints on the boundary of $D$, such that each of $D'$ and $D''$ contains at least one quarter of the boundary of $D$.*

With this we can get closer to Theorem 8.

**Proposition 12** – *Let $G = \langle\, a_1, \dots, a_m \mid R\,\rangle$ be a finite presentation and let $\ell_1, \ell_2$ be the minimal and maximal lengths of a relator in $R$.*

*Let $P$ be a class of van Kampen diagrams, such that any subdiagram of a diagram in $P$ lies in $P$.*

*Let $C, C' > 0$. Choose some $\varepsilon > 0$. Suppose that any van Kampen diagram $D$ in $P$ satisfies*

$$|\partial D| \geqslant C'\mathcal{A}(D)$$

*and that, for some $A \geqslant 50/(\varepsilon C')^2$, any van Kampen diagram $D$ in $P$ having boundary length at most $A\ell_2$ satisfies*

$$|\partial D| \geqslant C\mathcal{A}(D)$$

*Then any van Kampen diagram $D$ in $P$, with boundary length at most $7A\ell_2/6$, satisfies*

$$|\partial D| \geqslant (C-\varepsilon)\,\mathcal{A}(D)$$

**Proof** – Let $D$ be a van Kampen diagram in $P$, of boundary length between $A\ell_2$ and $7A\ell_2/6$. By the isoperimetry assumption for all diagrams we have $\mathcal{A}(D) \leqslant 7A\ell_2/6C'$.

By Lemma 11, we can partition $D$ into two diagrams $D'$ and $D''$, each of them containing at least one quarter of the boundary length of $D$. So we have $|\partial D'| \leqslant 3\,|\partial D|\,/4 + \ell_2(1 + 2\alpha \log(7A/6C')) \leqslant \ell_2\,(7A/8 + 1 + 2\alpha\log(7A/6C'))$ and likewise for $D''$.

Choose $A$ large enough (depending only on $C'$) so that $1 + 2\alpha\log(7A/6C') \leqslant A/8$. Then both $D'$ and $D''$ have boundary length at most $A\ell_2$. So by assumption we have

$$|\partial D'| \geqslant C\mathcal{A}(D') \ \text{ and } \ |\partial D''| \geqslant C\mathcal{A}(D'')$$

(note the occurrence of $C$ and not $C'$).

Now we choose $A$ large enough (again depending only on $C'$) so that $2 + 4\alpha\log(7A/6C') \leqslant \varepsilon A$ (if we remember that $\alpha \leqslant 1/C'$, taking $A = 50/(\varepsilon C')^2$ is enough). We have

$$\begin{aligned} |\partial D| &= |\partial D'| + |\partial D''| - 2\,|\partial D' \cap \partial D''| \\ &\geqslant |\partial D'| + |\partial D''| - \ell_2\,(2 + 4\alpha\log(7A/6C')) \\ &\geqslant C\,(\mathcal{A}(D') + \mathcal{A}(D'')) - \varepsilon A\ell_2 \\ &\geqslant (C-\varepsilon)\,\mathcal{A}(D) \end{aligned}$$

since $\mathcal{A}(D') + \mathcal{A}(D'') = \mathcal{A}(D) \geqslant |\partial D| \geqslant A\ell_2$. □

The last approximation to Theorem 8 is the following:

**Proposition 13** – *Let $G = \langle\, a_1, \dots, a_m \mid R\,\rangle$ be a finite presentation and let $\ell_1, \ell_2$ be the minimal and maximal lengths of a relator in $R$.*

*Let $P$ be a class of van Kampen diagrams, such that any subdiagram of a diagram in $P$ lies in $P$.*

*Let $C, C' > 0$. Choose some $\varepsilon > 0$. Suppose that any van Kampen diagram $D$ in $P$ satisfies*

$$|\partial D| \geqslant C'\mathcal{A}(D)$$

*and that, for some $K \geqslant 50/(\varepsilon^2 C'^3)$, any van Kampen diagram $D$ in $P$ having area at most $K\ell_2$ satisfies*

$$|\partial D| \geqslant C\mathcal{A}(D)$$

*Then any van Kampen diagram $D$ in $P$ satisfies*

$$|\partial D| \geqslant (C - 14\varepsilon)\,\mathcal{A}(D)$$

**Proof** – Set $A = C'K$. Let $D$ be a diagram in $P$ of boundary length at most $A\ell_2$. By the assumption on all diagrams, $D$ has area at most $A\ell_2/C' = K\ell_2$ so that by the assumption on small diagrams we have $|\partial D| \geqslant C\,\mathcal{A}(D)$. In particular, the assumptions of Proposition 12 are fulfilled.

So this proposition implies that diagrams $D$ in $P$ of area at most $7A\ell_2/6$ satisfy $|\partial D| \geqslant (C-\varepsilon)\,\mathcal{A}(D)$. This means that the assumptions of Proposition 12 are fulfilled with the new parameters $A_1 = 7A/6$, $\varepsilon_1 = \varepsilon(6/7)^{1/2}$ and $C_1 = C - \varepsilon$ instead of $A, \varepsilon, C$, and with the same $C'$ (these new parameters indeed satisfy $A_1 \geqslant 50/(\varepsilon_1 C')^2$).

So applying Proposition 12 again, we get that diagrams $D$ in $P$ of area at most $A_2 = A\ell_2(7/6)^2$ satisfy $|\partial D| \geqslant C_2\,\mathcal{A}(D)$ where $C_2 = C_1 - \varepsilon_1$.

By induction, we get that diagrams $D$ in $P$ of area at most $A\ell_2(7/6)^k$ satisfy

$$|\partial D| \geqslant \left( C - \varepsilon \sum_{i=0}^{k-1} (6/7)^{i/2} \right) \mathcal{A}(D)$$

and we conclude by the inequality $\sum_{i=0}^{\infty}(6/7)^{i/2} < 14$. □

**Proof of Theorem 8** – Applying Proposition 10 (which is allowed since $10^{50}\,(\ell_2/\ell_1)^3\,\varepsilon^{-2}\,C^{-3} \geqslant 10^{23}\,(\ell_2/\ell_1)\,C^{-2}$), we get that any van Kampen diagram $D$ in $P$ satisfies $|\partial D| \geqslant C'\,\mathcal{A}(D)$ where $C' = C\,(\ell_1/\ell_2)/10^{15}$. We conclude with Proposition 13 (where we replace $\varepsilon$ with $\varepsilon/14$). □

# Proof of Theorem 2

Now Theorem 2 is an easy consequence of Theorem 8 and already known facts about random groups. First, we recall the result from [Gro93] (see also [Oll04]) on diagrams of bounded size.

Suppose we are given a random presentation at density $d$, by reduced relators of length $\ell$.

**Theorem 14 (M. Gromov)** – *For every $\varepsilon > 0$ and every $K \in \mathbb{N}$, with overwhelming probability, every reduced van Kampen diagram with at most $K$ faces satisfies*

$$|\partial D| \geqslant (1 - 2d - \varepsilon)\,\ell\,|D|$$

Of course, the point is that the overwhelming probability is a priori not uniform in $K$.

**Proof** – We only have to change a little bit the conclusion of the proof in [Oll04], p. 613. It is proven there that if $D$ is a reduced van Kampen diagram involving $n \leqslant |D|$ distinct relators $r_1, \ldots, r_n$, with relator $r_i$ appearing $m_i$ times in the diagram (we can assume $m_1 \geqslant \ldots \geqslant m_n$), then there exist numbers $d_i$, $1 \leqslant i \leqslant n$ such that:

$$|\partial D| \geqslant (1 - 2d)\,\ell\,|D| + 2 \sum d_i(m_i - m_{i+1})$$

and such that the probability of this situation is at most $(2m)^{\inf d_i}$ ([Oll04], p. 613). In particular, for fixed $\varepsilon$, with overwhelming probability we can suppose that $\inf d_i \geqslant -\ell\varepsilon/2$.

If all $d_i$'s are non-negative, then we get $|\partial D| \geqslant (1-2d)\,\ell\,|D|$ as needed.

Otherwise, as $1 \leqslant m_i \leqslant |D|$ and $m_i \geqslant m_{i+1}$ we have $\sum d_i(m_i - m_{i+1}) \geqslant |D| \inf d_i$ and so

$$|\partial D| \geqslant (1 - 2d)\,\ell\,|D| + 2\,|D| \inf d_i \geqslant (1 - 2d - \varepsilon)\,\ell\,|D|$$

□

**Proof of Theorem 2** – Theorem 2 now is an immediate consequence of Theorem 14 and Theorem 8 (where the class $P$ is the class of all reduced diagrams). □

**Proof of Corollary 3** – For Corollary 3 we use the following proposition, which is only a weaker version, adapted to our vocabulary, of Lemma 3.11 of [Cha94]:

**Proposition 15** – *Suppose that a finite group presentation satisfies the following: for every reduced word $w$ representing the identity in the group, there exists a van Kampen diagram $D$ spanning $w$ with $|\partial D| \geqslant C\mathcal{A}(D)$. Let $\lambda$ be the maximal length of a relator in the presentation.*

*Then the group is $\delta$-hyperbolic with $\delta < 4\lambda/C$ (w.r.t. the metric defined by the generators in the presentation).*

Indeed, Lemma 3.11 of [Cha94] states that for some notion of area $\text{area}_{\text{Champetier}}$, the isoperimetric inequality $\text{area}_{\text{Champetier}}(D) \leqslant \alpha\,|\partial D|$ for van Kampen diagrams (actually for curves in a geodesic metric space) implies $\delta$-hyperbolicity with $\delta \leqslant 20\alpha$.

The notion of area used by Champetier (Definition 3.2 in [Cha94]) is different from $\mathcal{A}(D)$ as defined in this paper. However it is noted by Champetier that a curve of length $L$ has $\text{area}_{\text{Champetier}} \leqslant L^2/2\pi$. So for a van Kampen diagram $D$ we have

$$\text{area}_{\text{Champetier}}(D) \leqslant \sum_{f \text{ face of } D} \frac{|\partial f|^2}{2\pi} \leqslant \frac{\lambda}{2\pi} \sum_{f \text{ face of } D} |\partial f| = \frac{\lambda\mathcal{A}(D)}{2\pi}$$

Consequently the inequality $|\partial D| \geqslant C\mathcal{A}(D)$ implies that the Champetier assumption $\text{area}_{\text{Champetier}}(D) \leqslant \alpha\,|\partial D|$ holds with $\alpha = \lambda/(2C\pi)$, hence Proposition 15 noting that $20/2\pi < 4$.

Corollary 3 now follows from Theorem 2 and Proposition 15, choosing $\varepsilon$ small enough. □

**Proof of Corollary 4** – Corollary 4 is easy. Let $D = D_1 \cup_w D_2$. Since $|\partial D| \geqslant (1-2d-\varepsilon)\,\ell\,|D|$, the number of internal edges of $D$ is at most $(d+\varepsilon/2)\ell\,|D|$. So a fortiori $|w| \leqslant (d+\varepsilon/2)\ell\,|D|$. Now

$$\begin{aligned} |w| &\leqslant (d+\varepsilon/2)\,\ell\,|D| \leqslant \frac{d+\varepsilon}{1-2d-\varepsilon}\,|\partial D| \\ &= \frac{d+\varepsilon/2}{1-2d-\varepsilon}\,(|\partial D_1| + |\partial D_2| - 2\,|w|) \end{aligned}$$

and so

$$|w| \leqslant (d+\varepsilon/2)\ (|\partial D_1| + |\partial D_2|)$$

as needed. □

## Dehn's algorithm and Greendlinger's Property

We now turn to the proof of Theorem 6. Since the Greendlinger property is stronger than the Dehn algorithm, it suffices to prove the former for $d < 1/5$ and disprove the latter for $d > 1/5$.

**Greendlinger's Property for $d < 1/5$.** We begin by a lemma which is weaker in the sense that we do not ask for the boundary edges to be consecutive. We will then conclude by a standard argument.

**Lemma 16** – *For any $\varepsilon > 0$, with overwhelming probability, at density $d$ the following holds:*

*Let $D$ be a reduced van Kampen diagram with at least two faces. There exist two faces of $D$ each having at least $\ell(1 - 5d/2 - \varepsilon)$ edges on the boundary of $D$ (maybe not consecutive).*

Observe that when $d < 1/5$ this is more than $\ell/2$ (for small enough $\varepsilon$ depending on $1/5 - d$). This lemma is also valid at densities larger than $1/5$ but becomes trivial at $d = 2/5$.

**Proof of the lemma** – Let $D$ be a reduced van Kampen diagram with at least two faces.

First, note that it is enough to consider the case when $D$ is homeomorphic to a disk. Otherwise, decompose $D$ as the union of "filaments" and maximal parts homeomorphic to a disk. Adding or removing filaments does not change the property of a face having so many edges on the boundary of the diagram.

Let $f$ be a face of $D$ having the greatest number of edges on the boundary. Say $f$ has $\alpha\ell$ edges on the boundary. Suppose that any face other than $f$ has no more than $\beta\ell$ edges on the boundary. We want to show that $\beta \geqslant 1 - 5d/2 - \varepsilon$. So suppose that $\beta < 1 - 5d/2 - \varepsilon$. (The reader may find more convenient to read the following skipping the $\varepsilon$'s.)

Consider also the (maybe not connected, but this does not matter) diagram $D'$ obtained by removing face $f$ from $D$. We have $|\partial D'| = |\partial D| + \ell - 2\alpha\ell$.

By definition of $\alpha$ and $\beta$, and since $D$ is homeomorphic to a disk, we have $|\partial D| \leqslant \beta\ell(|D| - 1) + \alpha\ell$. Consequently $|\partial D'| \leqslant \beta\ell(|D| - 1) + \ell - \alpha\ell$.

But by Theorem 2, with overwhelming probability we can suppose that we have $|\partial D| \geqslant (1 - 2d - \varepsilon/2)\,\ell\,|D|$ and $|\partial D'| \geqslant (1 - 2d - \varepsilon/2)\,\ell\,|D'| = (1 - 2d - \varepsilon/2)\,\ell\,(|D| - 1)$. So combining these inequalities we get

$$(1 - 2d - \varepsilon/2)\,|D| \leqslant \beta(|D| - 1) + \alpha$$
$$(1 - 2d - \varepsilon/2)\,(|D| - 1) \leqslant \beta(|D| - 1) + 1 - \alpha$$

or, since we assumed by contradiction that $\beta < 1 - 5d/2 - \varepsilon$,

$$(1 - 2d - \varepsilon/2)\,|D| < (1 - 5d/2 - \varepsilon)\,(|D| - 1) + \alpha$$
$$(1 - 2d - \varepsilon/2)\,(|D| - 1) < (1 - 5d/2 - \varepsilon)\,(|D| - 1) + 1 - \alpha$$

which yield respectively

$$|D| < \frac{\alpha + 5d/2 - 1 + \varepsilon}{d/2 + \varepsilon/2} \tag{1}$$

$$|D| < \frac{d/2 + 1 - \alpha + \varepsilon/2}{d/2 + \varepsilon/2} \tag{2}$$

Either $\alpha \leqslant 1 - d - \varepsilon/4$ or $\alpha \geqslant 1 - d - \varepsilon/4$. In any case, one of (1) or (2) gives

$$|D| < \frac{3d/2 + 3\varepsilon/4}{d/2 + \varepsilon/2} < 3$$

(generally, a face having more than $(1-d)\ell$ on the boundary is the frontier at which it is more interesting to remove this face before applying Theorem 2).

The case $|D| \leqslant 2$ is easily treated by Theorem 2. So we get a contradiction, and the lemma is proven. □

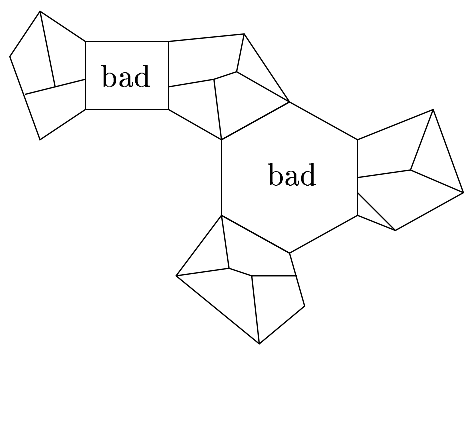
This somewhat obscure proof and the role of $1/5$ will become clearer in the next paragraph, when we will build a 3-face diagram for $d > 1/5$ with only one face having more than $\ell/2$ boundary edges.

Back to the proof of Greendlinger's Property for $d < 1/5$. If we are facing a diagram $D$ such that the intersection of the boundary of any face of $D$ with the boundary of $D$ is connected, then Lemma 16 provides what we want.

Now we apply a standard argument to prove that this case is enough. Suppose that some face of $D$ has a non-connected intersection with the boundary, having two (or more) boundary components, so that this face separates the rest of the diagram into two (or more) components. Call *good* a face having exactly one boundary component and *bad* a face with two or more boundary components (there are also internal faces, which we are not interested in).

First suppose that $D$ is homeomorphic to a disk (so that no single edge or vertex removal can disconnect it).

Decompose $D$ into bad faces and maximal parts without bad faces. Call such a maximal part *extremal* if it is in contact with only one bad face. It is clear that, if there exists some bad face, there are at least two such extremal parts.



To reach the conclusion it is sufficient to find in any extremal part a good face having more than $\ell(1-5d/2-\varepsilon)$ edges on the boundary. So let $f$ be a bad face in contact with an extremal part $P$ without bad faces.

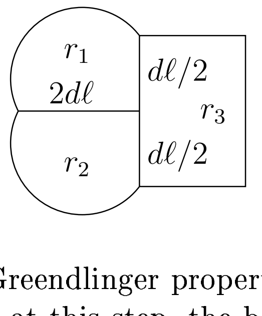
Consider the diagram $D' = P \cup f$. This diagram has no bad faces now, and so by Lemma 16 there are two faces in it having more than $\ell(1-5d/2-\varepsilon)$ consecutive edges on the boundary. One of these may be $f$, but the other one has to be in $P$ and so has more than $\ell(1-5d/2-\varepsilon)$ consecutive edges on the boundary of $D$ as well.

Now in the case $D$ is not homeomorphic to a disk, then the "filaments" (the edges/vertices the removal of which disconnects $D$) are treated the same way as bad faces in the previous argument.

**A counter-example for $d > 1/5$.** Here we show that the presentation does not satisfy the Dehn algorithm as soon as $d > 1/5$.

Fix some $\varepsilon > 0$. We can with overwhelming probability find two relators $r_1, r_2$ sharing a common subword $w$ of length $(2d-\varepsilon)\ell$. Once those are chosen, let $x$ be the subword of length $(d-\varepsilon)\ell$ of the boundary of the diagram $r_1 \cup_w r_2$ occurring around some endpoint of the $w$-gluing and having length $(d-\varepsilon)\ell/2$ on each side of this endpoint (see picture below). (When $d > 2/5$ there is less than this left on the boundary of $r_1 \cup_w r_2$; but the situation is even easier at larger densities and so we leave this detail aside).

At density $d$, subwords of length $(d-\varepsilon)\ell$ of the relators exhaust all reduced words of length $(d-\varepsilon)\ell$. So it is possible to find a relator $r_3$ gluing to $r_1 \cup_w r_2$ along $x$. After this operation $r_1$ and $r_2$ each have less than $1-(2d-\varepsilon)\ell-(d/2-\varepsilon/2)\ell = (1-5d/2+3\varepsilon/2)\ell$ of their length on the boundary of the new diagram (see picture below), which is less than $\ell/2$ when $d > 1/5$, for small enough $\varepsilon$. Compare Lemma 16 — which is thus sharp.

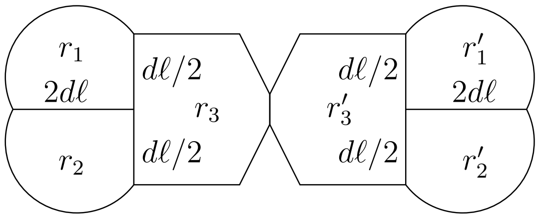



This diagram violates the Greendlinger property (but not yet the Dehn algorithm). Note for later use that at this step, the boundary length of the diagram so obtained is $(3-6d+4\varepsilon)\ell$. This is the smallest possible value compatible with Theorem 2, up to the $\varepsilon$'s.

But (thanks to the $\varepsilon$'s) this will not only happen once but arbitrarily many times as $\ell \to \infty$, so we can find another independent triple of relators $(r_1', r_2', r_3')$ giving rise to the same configuration.

Now if $r_3$ and $r_3'$ share only a single letter in the region of length $\ell/5$ opposite to the position where they glue to $r_1 \cup_w r_2$ (resp. $r_1' \cup_{w'} r_2'$) (and this happens all the time thanks to the law of large numbers), then we can form a diagram in which $r_3$

and $r'_3$ become faces having no more than $\ell/2$ consecutive edges on the boundary (they are bad faces in the terminology of the previous proof). So if $d > 1/5$, no face of this diagram has more than $\ell/2$ consecutive edges on the boundary (although the two bad faces have more than $\ell/2$ non-consecutive boundary edges).




This is not enough to disprove the Dehn algorithm: this algorithm only demands that for any reduced word representing $e$, there exists *some* van Kampen diagram with the boundary face property. There could exist another van Kampen diagram $D'$ with the same boundary word as the diagram $D$ above, in which some face would have more than $\ell/2$ consecutive edges on the boundary. So let $r_4$ be this face. Since $D$ and $D'$ have the same boundary word, we can glue $r_4^{-1}$ to the previous diagram $D$ to get a new diagram $D''$ with 7 faces; since $r_4$ has more than half of its length on the boundary of $D'$ we have $|\partial D''| < |\partial D|$.

Either $D'$ is reduced or $r_4$ is equal to some relator $r_i$ already present in the diagram.

In the former case, we get that $|\partial D| = (3-6d+4\varepsilon)\ell\times 2-2 = 6(1-2d)\,\ell+8\varepsilon\ell-2$. Since $|\partial D'| < |\partial D|$ we get $|\partial D'| < 6(1-2d)\,\ell + 8\varepsilon\ell - 2$. But by Theorem 2, for any $\varepsilon'$ we have $|\partial D'| \geqslant 7(1-2d-\varepsilon')\ell$, which is a contradiction for small enough values of $\varepsilon$ and $\varepsilon'$.

In the latter case, this means that we can glue a copy of $r_i^{-1}$ along $r_i$ on the boundary of the diagram $D$ along more than $\ell/2$ edges. But, since the $r_i$ included in $D$ has no more than $\ell/2$ consecutive edges on the boundary, this means that before gluing $r_i^{-1}$ we could have folded some letters of $r_i$ with neighbouring letters in the boundary of $D$. This is excluded if we assume (as we can always do) that the boundary of $D$ is reduced.

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