Cayley graphs containing expanders, after Gromov

Yann Ollivier

13th October 2003

Abstract

We give the sketch of a combinatorial proof of the construction by Gromov of a group whose Cayley graph contains a family of expanders.

Combining the methods in [Oll1] and [Oll2], it is possible to give a proof of the construction invented by M. Gromov in [G] that there is an infinite group whose Cayley graph contains (in some quasi-isometric sense) a family of expanders. We only give the main steps of the proof, as our goal is to illustrate our techniques and not to re-prove known theorems.

This text is a natural sequel to [Oll2], and also heavily relies on sections 5.1.1, 6.2 to 6.6 and Appendix A of [Oll1].

The main lines of the argument of [G] are also explained in [Gh].

1 Quotients of hyperbolic groups by labelled graphs

We give here a statement of a theorem generalizing the one stated in [Oll2], together with a sketch of proof.

We use the terminology of [Oll2]: Γ is a graph labelled with the generators $a_1^{\pm 1}, \ldots, a_m^{\pm 1}$ of some group G_0, and we want to study the quotient of G_0 by the words read on cycles of the graph.

In this more complex situation, an *ε-piece with respect to G_0* is a couple of words (w_1, w_2) embedded in Γ (not necessarily distinct) together with a couple of words (δ_1, δ_2), such that $|\delta_1| + |\delta_2| \leqslant \varepsilon(|w_1| + |w_2|)$ and such that $w_1 = \delta_1 w_2 \delta_2$ in G_0.

Again we have to eliminate trivial cases: for example, if the word uwv is embedded in Γ, then $((uw, wv), (u, v^{-1}))$ is a trivial piece. Generally, a *trivial piece* is a piece $((w_1, w_2), (\delta_1, \delta_2))$ such that there is a path in Γ joining the beginning of w_1 to the beginning of w_2, labelling a word equal to δ_1 in G_0.

The *length* of a piece is defined as $\max(|w_1|, |w_2|)$.

We will say that a group is *aspherical* if it admits a presentation with no spherical van Kampen diagram (with the convention of [Oll1] for van Kampen diagrams). This implies asphericity of the Cayley complex for this presentation, and thus geometric and cohomological dimension at most 2 hence torsion-freeness.

The theorem is as follows.

Theorem 1 (M. Gromov) – *Let Γ be a labelled graph. Let R be the set of words read on all cycles of Γ (or on a generating family of cycles). Let g be the girth of Γ.*

Let G_0 be an aspherical non-elementary hyperbolic group. Let $\varepsilon > 0$, $C > 0$ and $\lambda < 1/6$.

Suppose that g is large enough (depending on $G_0, \varepsilon, C, \lambda$), that $\operatorname{diam}\Gamma \leqslant Cg$ and that the length of the longest non-trivial ε-piece with respect to G_0 is at most λg.

Suppose that there exists a constant $A > 0$ such that any word w embedded in Γ of length at least $(1-6\lambda)g/2$ satisfies $\|w\|_{G_0} \geqslant A(|w|-L)$ for some $L \leqslant (1-6\lambda)g/2$.

Then the group $G = G_0/\langle R\rangle$ is hyperbolic, aspherical and infinite. The radius of injectivity of the quotient map $G_0 \to G$ is at least $Ag/4$, and the natural application from the labelled graph Γ to the Cayley graph of G is a $(2/A, 2AL)$ quasi-isometry. Moreover, the Euler characteristic of G is that of G_0 plus the rank of the fundamental group of Γ.

Computation of the Euler characteristic ensures for example that if the rank of the fundamental group of Γ is greater than the number of generators, then G is non-elementary (the probabilistic argument for non-elementarity given in [Oll1] is not available here). Indeed, since G is torsion-free, if it is elementary it is either $\{e\}$ or $\mathbb{Z}$, whose Euler characteristic is 1 and 0 respectively, which are excluded when the number of relators is greater than the number of generators (using asphericity).

The apparently easiest (though the details may be equally tedious) proof, with λ small enough instead of $\lambda < 1/6$, uses a geometrized version of the Eilenberg-MacLane space for a group as described in Section 1.7 of [G]. We propose here a combinatorial proof. We only indicate how to put together the ideas from [Oll1] and [Oll2] and do not repeat the parts of the argument that can be directly transposed.

Proof (elements thereof) –

Let R be the set of words read on the cycles of Γ (or on a generating set of cycles).

One has to consider a van Kampen diagram D for the quotient, which contains new relators from R and old relators from a presentation of G_0. Take as a generating set of the cycles of Γ all cycles of length at most $3\operatorname{diam}\Gamma$. As the diameter of Γ is bounded by C times the girth g of Γ, the ratios of the lengths of these cycles lie between $1/3C$ and $3C$. This is crucial as it is what allows to apply the local-to-global hyperbolic principle (Cartan-Hadamard-Gromov-Papasoglu theorem) described in Appendix A of [Oll1].

Here we will work exactly as in Sections 6.2 and 6.3 of [Oll1]. First, proceed as in [Oll2] to group the new relators of D that share the same edges in D that they share in Γ (that is, group them into maximal parts which can be lifted to Γ). We now reason at the scale of these blocks. The old relators of D form a thin layer around the new ones, and so it is possible to decompose them into strips which contain two long sides w_1, w_2 on the boundary of new relators, and two short sides (w.r.t. g) δ_1, δ_2, such that $w_1 = \delta_1 w_2 \delta_2$ in Γ. This is where pieces come in. Trivial pieces do not matter, as a trivial piece precisely means that the two neighbouring relators are glued along edges they already share in Γ, and so it is possible to suppress the trivial piece and group the two relators.

Now use the assumption on the length of pieces and the small cancellation theory as presented in [Oll2] to conclude that if D is a strongly reduced van Kampen diagram, with $D = D' \cup D''$ with the old relators in D' and the new ones in D'', with D' minimal, with at most K new relators, and such that the parts of D that lift to Γ are minimal, then D satisfies an isoperimetric inequality of the form

$$|\partial D| \geqslant \alpha_1 g \, |D''| + \alpha_2 \, |D'|$$

where α_1 comes from the small cancellation theory (see Remark 7 of [Oll2]), and α_2 comes from the isoperimetric inequality in G_0. (Compare Proposition 31 in [Oll1].)

Conclude using the hyperbolic local-to-global principle.

As for the radius of injectivity of $G_0 \to G$ and the quasi-isometricity of the map $\Gamma \to G$, the factors 2 instead of what would be expected (which can be reduced to $(1 + o(1)_{g\to\infty})$) come from the fact that van Kampen diagrams for the quotients contain some strips of old relators which can slightly decrease the boundary length (which stays essentially A times the boundary length of the new relators within). One has to be cautious about this, because the local-to-global principle worsens the constants: one has to take K large enough so that, when establishing quasi-isometricity of $\Gamma \to G$ (which involves looking at words of length at most Cg), we can work only on diagrams with at most K new cells, for which we know decent constants (by applying directly the small cancellation theory), instead of diagrams whose isoperimetry was established through the local-to-global principle.

Computation of the Euler characteristic is immediate given that the cohomological dimension is 2. □

2 Application to random labellings

We now show that a random labelling of the graphs used in [G] very probably satisfies the assumptions of the above theorem. We recall the properties of these graphs.

Definition 2 – *Let V_{gj}, $g, j \in \mathbb{N}$, be a family of graphs. It is said to be* good for random quotients *if the following holds:*

- $\operatorname{girth} V_{gj} = g$
- $\operatorname{diam} V_{gj} \leqslant 100g$
- $|V_{gj}| \geqslant 2^{g/j}$

- *The balls $B(v, g/2) \subset V_{gj}$, with $v \in V_{gj}$, satisfy $|B(v, g/2)| \leqslant 10(2^{g/2j})$.*

Note that, by multiplicativity of the growth of balls, the control on $|B(v,r)|$ immediately extends to $r \leqslant \operatorname{diam} V_{gj}$, namely $|B(v,r)| \leqslant 2^{100r/j+800}$. Of course, 100, 2 and 10 can be replaced by other constants.

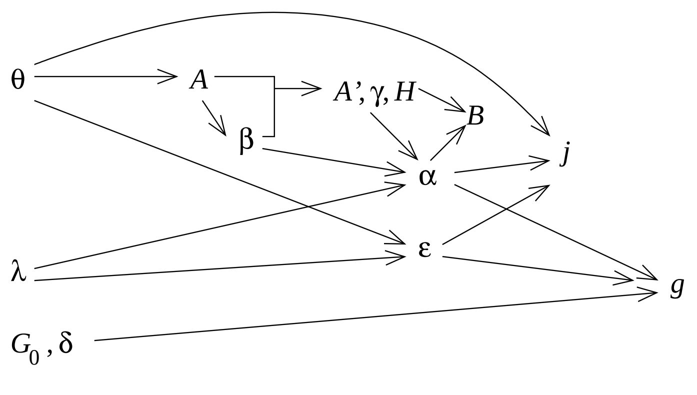
Proposition 3 (M. Gromov) – *Let $\theta < 1$ and $0 < \lambda < 1/6$. There exists $\varepsilon > 0$, $j_0 \in \mathbb{N}$, $A > 0$ and $B < 1$ such that for any torsion-free hyperbolic group G_0 generated by $a_1^{\pm 1}, \ldots, a_m^{\pm 1}$ with $m \geqslant 2$ with gross cogrowth at most θ, a random labelling of a good graph for random quotients satisfies the assumptions of Theorem 1, provided $j \geqslant j_0$ and g large enough (depending on G_0) i.e.:*

- *The longest non-trivial ε-piece has length at most λg;*
- *Setting $L = B(1-6\lambda)g/2$, any word w embedded in Γ satisfies $\|w\|_{G_0} \geqslant A(|w| - L)$.*

Proof –

Suppose Γ has been given a random labelling.

There are many variables involved in the proof. In order to help the understanding of what follows, we include a dependency graph of the variables used in the proof. Variables on the right are set last depending on variables on the left.



Let B_n denote the random element of G_0 obtained after a n-step random walk in G_0. On B_n we will use the probability estimates from Section 5.1.1 of [Oll1]. Estimates therein are asymptotic; in order to get exact inequalities, we have to suppose that the lengths of the words involved are large enough (depending on G_0) and, moreover, we introduce a factor 1/2 in the exponents to compensate for the limit: for example, instead of $\Pr(B_n = e) \lesssim (2m)^{-(1-\theta)n}$ we will use $\Pr(B_n = e) \leqslant (2m)^{-(1-\theta)n/2}$ for n large enough.

Let p be a path embedded in Γ, of length between αg and $g/2$ for some small $\alpha > 0$ to be specified later (depending on G_0 and λ but not on g). As the girth of Γ is g, p does not run twice through the same point of Γ, so the letters of the word w labelled by p are chosen independently and the law of w is the law of $B_{|w|}$.

Let $\|w\|$ denote the norm of w in G_0. After Proposition 17 of [Oll1], the probability that $\|w\| \leqslant \frac{1-\theta}{2\theta}|w|$ is less than $(2m)^{-(1-\theta)|w|/4}$ provided that $|w|$ is large enough depending on G_0 (this will be the case since $|w| \geqslant \alpha g$).

This is for a given path in Γ. A path of length at most $g/2$ is uniquely defined by two points in Γ at distance at most $g/2$. The number of points in Γ is at most $2^{100g/j+800}$. The number of points at distance at most $g/2$ from a given point is at most $2^{g/2j+4}$. So the number of possible paths p is at most $2^{100,5g/j+804}$. The probability that for some path p, we have $\|w\| \leqslant \frac{1-\theta}{2\theta}|w|$ is thus less than $2^{100,5g/j+804}(2m)^{-\frac{1}{2}(1-\theta)|w|/2}$. As $|w| \geqslant \alpha g$, for j large enough (depending on α, θ and 804 but not on g), we have $100,5g/j < \frac{1}{2}(1-\theta)\alpha g/2$ for any g and so this probability decreases exponentially when g grows to infinity.

So for g large enough, with probability tending to 1 as $g \to \infty$, any word embedded in Γ, of length between αg and $g/2$, satisfies $\|w\| \geqslant A|w|$ with $A = \frac{1-\theta}{2\theta}$.

In particular, any word embedded in Γ (without restriction of length) is a $(1/A, \alpha g, g/2)$ local quasi-geodesic (in the notation of [GH]). But any local quasi-geodesic is a global quasi-geodesic (with some loss in the constants). More precisely, suppose (as proven in [GH]) that any $(1/A, 1, \beta)$ local quasi-geodesic in a 1-hyperbolic space, with β large enough, is a $(1/A', \gamma)$ global quasi-geodesic lying at Hausdorff distance at most H from the geodesic joining its endpoints. (The values of A', β and γ depend on θ but not otherwise on G_0.) In our case we have a $(1/A, \alpha g, g/2)$ local quasi-geodesic. Scale by $1/\alpha g$. If $\alpha g \geqslant \delta$ and if $\alpha < 1/2\beta$ (where δ is a hyperbolicity constant for G_0), the scaling gives a $(1/A, 1, \beta)$ local quasi-geodesic in a 1-hyperbolic space. Hence (after scaling back by αg) any word embedded in Γ is a $(1/A', \gamma\alpha g)$ global quasi-geodesic lying at distance at most $\alpha g H$ from some geodesic.

For the quasi-embedding assumption of the theorem to be satisfied, take $B = \gamma\alpha$. It is thus enough to take α such that $\gamma\alpha g < (1-6\lambda)g/2$ (depending only on θ and λ, as does γ), and then g large enough (depending on everything) so that $\alpha g \geqslant \delta$ and that the probabilistic estimates based on θ hold.

For the small cancellation condition, suppose we are given a non-trivial ε-piece $((w_1, w_2), (\delta_1, \delta_2))$ as defined above, with $w_1 = \delta_1 w_2 \delta_2$ in G_0, with $|\delta_{1,2}| \leqslant \varepsilon g$ and $\max(|w_1|, |w_2|) \geqslant \lambda g$.

First, we will shorten w_1 and/or w_2 as to get $\max(|w_1|, |w_2|) = \lambda g$. Indeed, suppose $|w_1| > \lambda g$ and let w_1' be the initial subword of w_1 of length λg. Let Δ_1, Δ_2 be the geodesic segments (we reason in G_0) joining the ends of w_1, w_2, respectively. As w_1 lies at Hausdorff distance $\gamma\alpha g$ from Δ_1, there is a point x on Δ_1 at distance at most $\gamma\alpha g$ from the endpoint of w_1'. As Δ_1 and Δ_2 lie at distance εg from each other, let y be a point of Δ_2 at distance at most εg from x, and let w_2' be an initial subword of w_2 such that the endpoint of w_2' lies at distance at most $\gamma\alpha g$ from y. Now if δ_2' is a word joining the endpoints of w_1' and w_2', we have $w_1' = \delta_1 w_2' \delta_2'$ with $|\delta_2'| \leqslant (\varepsilon + 2\gamma\alpha)g$. After this we have $|w_1'| = \lambda g$; if still $|w_2'| > \lambda g$ we use the same trick again.

Finally, we can suppose that $\max(|w_1|, |w_2|) = \lambda g$ and that $|\delta_{1,2}| \leqslant (\varepsilon + 4\gamma\alpha)g$ (namely, if there exists a longer piece, then a smaller one exists as well).

Now for the evaluation of the probability of the existence of some such piece.

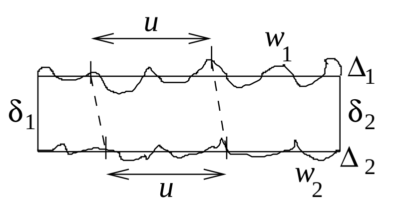
Since the lengths of w_1 and w_2 are smaller than half the girth of Γ, w_1 and w_2 are determined by their endpoints. There are at most $2^{100g/j+800}$ points in Γ, so the number of positions for w_1 and w_2 in Γ is at most $2^{400g/j+3200}$.

We are going to show that, given the position of w_1 and w_2 in Γ, the probability that they form a non-trivial piece behaves at most like $(2m)^{-(1-\theta)(|w_1|+|w_2|)/2}$. As $\max(|w_1|, |w_2|) = \lambda g$ this is at most $(2m)^{-(1-\theta)\lambda g/2}$. So if j is large enough, namely $j > 200/\lambda(1-\theta)$, the total probability that there exists a piece will be exponentially small in g.

First, suppose that w_1 and w_2 do not intersect (as parts of Γ). Then the random letters making up w_1 and w_2 are independent, and after Proposition 22 of [Oll1], the probability of such a situation is less than $(2m)^{2\varepsilon g-(1-\theta)(|w_1|+|w_2|)/2}$ provided $|w_1|+|w_2|$ is large enough depending on G_0, which holds since $\max(|w_1|, |w_2|) > \lambda g$, provided g is large enough. So this probability is at most $(2m)^{-(2\varepsilon+8\gamma\alpha)g-(1-\theta)\lambda g/2}$, and if ε and α are small enough (depending on θ and λ), this probability will be exponentially small.

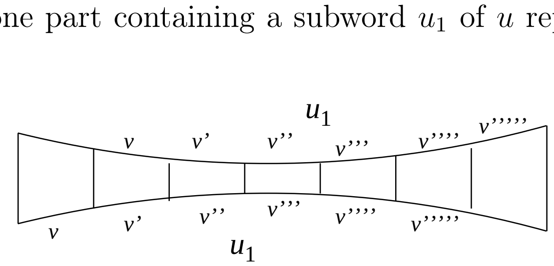
Second, suppose that w_1 and w_2 intersect (in Γ). We first prove that the intersection is connected (is made of a single subword). Suppose indeed that there are two points in the intersection. As the length of w_1 and w_2 is less than half the girth of Γ, the two subwords of w_1 and w_2 joining these two points are the same.

So let u be this intersection. Apart from the letters in u, the random letters in w_1 and w_2 are independent. Let Δ_1, Δ_2 be geodesic segments joining the endpoints of w_1, w_2 respectively; they lie at Hausdorff distance at most $\gamma\alpha g$ from w_1 and w_2 and at distance at most $(\varepsilon + 4\gamma\alpha)g$ from each other.



Set $e = \varepsilon + 6\gamma\alpha$. Call two points $x \in w_1$, $y \in w_2$ *facing* if their distance is at most eg. For any point on w_1 or w_2 there is a point facing it. If no point of $u \subset w_1$ faces some point of $u \subset w_2$, cutting the figure into two parts at some facing points in between allows to reason on each part separately as above.

Otherwise, reasoning as in [Oll1] (section "Elimination of pieces", where our *pieces* here are a kinds of *translators* there), it is possible to cut the figure into (at most five) parts at facing points and either get parts without repetition or, in the worst case, isolate one part containing a subword u_1 of u repeated on both sides with some shift:



The contribution of the parts not containing u_1 (there are at most four of them, as constructed in [Oll1]) to the probability of the situation is of the form

$(2m)^{8eg-(1-\theta)(|w_1|+|w_2|-2|u_1|)/2}$ where $e = \varepsilon + 6\gamma\alpha$. So if ε and $\gamma\alpha$ are small enough (depending on θ), this is exponentially small in $|w_1| + |w_2| - 2\,|u_1|$.

Now for the part containing u_1. If the shift is bigger than $g\sqrt{e}$, then cutting this figure into $|u_1|\,/g\sqrt{e}$ parts at some facing points allows to reason separately on each figure; evaluating the probability of each part separately and multiplying, the probability of this situation is less than $(2m)^{2eg\cdot(|u_1|/g\sqrt{e})-(1-\theta)|u_1|}$. When multiplied by the probability of the other parts above, this gives a probability which is exponentially small in $|w_1| + |w_2|$.

If the shift is smaller than $g\sqrt{e}$, then we are in the situation described by Axiom 4 of [Oll1] (here we use torsion-freeness) and the probability of such an event is at most $(2m)^{2g\sqrt{e}-(1-\theta)|u_1|}$ (the small sides of the figure are non-trivial since the piece was supposed to be non-trivial). Once multiplied by the probability coming from the other parts, the probability of the whole situation is again exponentially small in $|w_1| + |w_2|$.

This ends the proof. □

In the proof above, L/g can even be made arbitrarily small.

3 Altogether

The scheme of the proof is now the following: We start with an aspherical, hyperbolic group G_0.

We are going to take successive random quotients of G_0. In order to do so we will crucially need a control on the gross cogrowth θ of all these quotients. The simplest way to do so is to take G_0 having property T. This ensures that the gross cogrowth (w.r.t. a given generating set) of all infinite quotients of G_0 is controlled by a Kazhdan constant for G_0 w.r.t. this generating set. (Indeed, gross cogrowth is linked to the spectral radius of the averaging operator of the left regular representation of the group.)

A simple way to get a hyperbolic group of dimension 2 and with property T is simply to take a random group, as described in [Z].

Adapting the techniques from [Oll3], it may even be possible to avoid property T and to show directly that the gross cogrowth of the successive random quotients are close to that of G_0.

Now we can state

Theorem 4 (M. Gromov) – *Let V_{gj} be a family of graphs which is good for random quotients, in the sense above. There exists an infinite group G_∞, an integer j, a constant $a > 0$ and an infinite non-decreasing sequence g_k of integers such that for any k, there exists a map of graphs $\varphi_k : V_{g_k j} \to G_\infty$ such that for all $v, v' \in V_{g_k j}$ we have*

$$a\,(\mathrm{dist}(v, v') - g_k/4) \leqslant \mathrm{dist}(\varphi_k(v), \varphi_k(v')) \leqslant \mathrm{dist}(v, v')$$

(The distance in G_∞ refers to some finite generating set.) To appreciate the $g/4$, remember that g is the girth of V_{gj}. In particular, this implies that

$$\lim_k \max_{x\in G_\infty} \frac{\left|\varphi_k^{-1}(x)\right|}{\left|V_{g_k j}\right|} = 0$$

which is the assumption needed in [HLS] to provide counter-examples to the Baum-Connes conjecture. Indeed, since φ_k is a $(1/a, ag_k/4)$ quasi-isometry, any point of $\varphi_k^{-1}(x)$ is at distance at most $g_k/4$ from any other one. Then, using the control on the balls in the definition above of good graphs for random quotients, we get $\left|\varphi_k^{-1}(x)\right| \leqslant 10(2^{g_k/2j})$. But by assumption $\left|V_{g_k j}\right| \geqslant 2^{g_k/j}$ hence the claim since $g_k \to \infty$. So if (as explained in Section 3.13 of [G]) the V_{gj}, for fixed j, are taken to be a family of expanders (which we do not need here), the conditions in [HLS] are fulfilled.

The quasi-isometry coefficient $1/4$ (in $g_k/4$) can even be made arbitrarily small.

Proof – Let G_0 be an aspherical, non-elementary hyperbolic group with property T (e.g. a random group as in [Z]), of rank 2.

Let θ_0 be an upper bound for the gross cogrowth of infinite quotients of G_0 (arising from a Kazhdan constant). Let $\lambda = 1/12$.

Let j be the j mentioned in Proposition 3 applied to θ_0 and λ, and let a be the A in the same proposition. Let g_0 be large enough (depending on G_0) to ensure that this proposition can be applied to $V_{g_0 j}$.

Apply Proposition 3, then Theorem 1 to G_0 and a random labelling of $V_{g_0 j}$. The quotient G_1 thus defined is a hyperbolic group enjoying the same properties as G_0: its gross cogrowth is at most θ_0, it is aspherical (hence torsion-free), hyperbolic, of rank 2.

Moreover, torsion-freeness implies that if it is elementary, it is either $\mathbb{Z}$ or $\{e\}$. As a T-group it cannot be $\mathbb{Z}$, and since some non-trivial graph embeds in its Cayley graph it cannot be $\{e\}$.

Therefore, it is possible to repeat the argument with some g_1 large enough (depending of course on G_1), but, most importantly, with the same j (which depends only on θ_0), to get a second group G_2 enjoying the same properties, obtained by a random quotient from the graph $V_{g_1 j}$.

By induction, one can obtain an infinite sequence of successive infinite groups $G_0 \to G_1 \to G_2 \to \ldots$ The limit group G_∞ is infinite as a limit of infinite groups on the same finite set of generators.

Moreover, by the conclusions of Theorem 1, at each stage we get a natural map $\psi_{k-1} : V_{g_{k-1} j} \to G_k$ which is an $(a, ag_{k-1}/4)$ quasi-isometry (a depends only on θ_0). As the quotient maps $G_k \to G_{k+1}$ have injectivity radius at least $ag_k/4$ (by still another conclusion of Theorem 1), and as the diameter of $V_{g_{k-1} j}$ is at most $100 g_{k-1}$, as soon as $g_k > 400 g_{k-1}/a$ the image $\psi_{k-1}(V_{g_{k-1} j})$ injects into all subsequent groups $G_{k+1}, G_{k+2} \ldots$ and hence into G_∞. So the compound map $\varphi_{k-1} : V_{g_{k-1} j} \to G_\infty$ are indeed $(a, ag_{k-1}/4)$ quasi-isometries. $\square$

Thanks to Pierre Pansu for having insisted on me that I should study this construction of Gromov, and for comments on the manuscript. Thanks to Thomas Delzant and Étienne Ghys for sharing their own understanding of the thing.

References

[G] M. Gromov, *Random Walk in Random Groups*, Geom. Funct. Anal. **13** (2003), No. 1, 73–146.

[Gh] É. Ghys, , *Groupes aléatoires*, séminaire Bourbaki **916** (2003).

[GH] É. Ghys, P. de la Harpe, *Sur les groupes hyperboliques d'après Mikhael Gromov*, Progress in Math. **83**, Birkhäuser (1990).

[HLS] N. Higson, V. Lafforgue, G. Skandalis, *Counterexamples to the Baum-Connes conjecture*, Geom. Funct. Anal. **12** (2002), No. 2, 330–354.

[LS] R.C. Lyndon, P.E. Schupp, *Combinatorial Group Theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete **89**, Springer (1977).

[Oll1] Y. Ollivier, *Sharp phase transition theorems for hyperbolicity of random groups*, to appear in Geom. Funct. Anal. Preprint available on ArXiV: math.GR/0301187

[Oll2] Y. Ollivier, *On a small cancellation theorem of Gromov*, preprint available on ArXiV: math.GR/0310022

[Oll3] Y. Ollivier, *Growth and cogrowth are stable under a random quotient*, in preparation.

[Z] A. Żuk, *Property (T) and Kazhdan constants for discrete groups*, Geom. Funct. Anal. **13** (2003), No. 3, 643–670.