Multiscale cosmological dynamics

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Abstract

The recently developed mean field theory of relativistic gravitation predicts the emergence of an "apparent matter" field at large scales describing the net effect of small-scale fluctuations on the large-scale dynamics of the universe. It is found that this so-called back reaction effect is much stronger for gravitational waves than for matter density fluctuations. At large scales, gravitational waves behave like radiation and, for them, the perturbative effect scales as the squared relative amplitude times squared frequency. In particular, a bath of gravitational waves of relative amplitude 10^{-5} and frequency 10^{-12} Hz would not be directly detectable by today's technology but would generate an effective large-scale radiation of amplitude comparable to the unperturbed matter density of the universe.

1 Introduction

Multiscale systems are characterized by intricate dynamics which couple several different time or space scales. Nevertheless, in many instances it is

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possible to obtain an effective dynamics governing the evolution of a multiscale system on a larger scale by averaging the full exact dynamics on smaller scales. Examples range from economics [1] to biophysics [2] and include turbulence [3] and quantum field theory at both vanishing and finite temperature [4].

This article deals with relativistic gravitating systems. These are described, at the classical level, by general relativity [5], and are multiscale because Einstein's theory is strongly non-linear. The largest gravitating system is the universe and its large-scale description is the traditional object of cosmology. It is now well established [6] that the universe is, on large scale, expanding in an homogeneous and isotropic manner. Several authors [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17] have recently argued that small-scale fluctuations around this large-scale expansion may, by non-linearity, contribute substantially to the large-scale energy repartition generating the expansion. This article investigates the importance of this so-called "back reaction" effect on dust universes perturbed by background gravitational waves and small matter density fluctuations. Our main conclusion is that matter density perturbations produce a negligible back reaction but that, on the other hand, background gravitational waves may generate a large-scale energy density comparable to the energy density of dust. Implications of these results for physical cosmology, including the dark energy problem, are also discussed.

2 Mean field theory

2.1 Notation

In this article, the metric has signature $(+,-,-,-)$. We shall use mixed components T_μ^ν of the stress-energy tensor; with this signature, for a perfect fluid at rest with density ρ and pressure p we have $T_0^0 = \rho$ and $T_i^i = -p$.

2.2 General framework

Averaging classical gravitational fields necessitates a mean field theory of general relativity. Such a theory has been introduced in [7, 8]; we include a brief overview here for completeness. Some applications to black hole physics are presented in [18, 19, 20].

Let $\mathcal{M}$ be a fixed manifold, let Ω be an arbitrary probability space and let $g(\omega)$ be a Lorentzian metric on $\mathcal{M}$ depending on the random parameter ω. Each pair $\mathcal{S}(\omega) = (\mathcal{M}, g(\omega))$ represents a physical space-time depending on the random parameter $\omega \in \Omega$. For example, $g(\omega)$ may represent a gravitational wave of random phase and wave vector around a given reference space-time.

With each space-time $\mathcal{S}(\omega)$ are associated the Einstein tensor $G(\omega)$ of the metric $g(\omega)$, and a stress-energy tensor $\mathcal{T}(\omega)$, satisfying the Einstein equation

$$G(\omega) = 8\pi\mathcal{T}(\omega). \tag{1}$$

As shown in [7], such a collection of space-times can be used to define a single mean space-time $(\mathcal{M}, \bar{g})$ representing the average, "macroscopic" behavior of these random space-times. The metric $\bar{g}$ is the average of the metrics $g(\omega)$; thus, at every point P of $\mathcal{M}$,

$$\bar{g}(P) = \langle g(P, \omega) \rangle \, . \tag{2}$$

where the brackets on the right-hand side denote an average over the random parameter ω. If we think of $g(\omega)$ as a reference, "macroscopic" metric perturbed by small random contributions then $\bar{g}$ will represent the average metric, where the fluctuations have been smoothed out but with the same macroscopic behavior.

The metric $\bar{g}$ defines an Einstein tensor $\bar{G}$ for the mean space-time. However, since the expression for the Einstein tensor as a function of the metric is non-linear, this Einstein tensor and the average stress-energy tensor will, in general, *not* be related by the Einstein equation:

$$\bar{G} \neq 8\pi \langle \mathcal{T}(\omega) \rangle \, ,$$

so that physical measurements attempting to relate the mean space-time to its average matter content would yield to a violation of the Einstein equation. This would not happen in a Newtonian setting since, then, the relation between field and matter is linear and thus is unchanged under averaging.

To enforce validity of the Einstein equation for the mean space-time, it is thus necessary to introduce a new term

$$\mathcal{T}^{\mathrm{app}}{}^{\nu}_{\mu} = \bar{G}^{\nu}_{\mu}/8\pi - \left\langle \mathcal{T}^{\nu}_{\mu}(\omega) \right\rangle , \tag{3}$$

so that the stress-energy tensor of the mean space-time can be described as the sum of the average stress-energy tensor $\langle \mathcal{T}(\omega) \rangle$ appearing in the averaged space-times, and of this new term $\mathcal{T}^{\mathrm{app}}$. This generally non-vanishing tensor field can be interpreted as the stress-energy tensor of an "apparent matter" in the mean space-time. Apparent matter describes the cumulative non-linear effects of the averaged-out small-scale fluctuations on the large-scale behaviour of the mean gravitational field.

In particular, even the vanishing of $\mathcal{T}(\omega)$ for all ω does not necessarily imply the vanishing of $\bar{\mathcal{T}}$. The mean stress-energy tensor $\bar{\mathcal{T}}$ can therefore be non-vanishing in regions where the unaveraged stress-energy tensor actually vanishes. We will see that this happens, for instance, with gravitational waves.

The justification and implications of this averaging procedure are discussed at length in [7, 8], together with an extension including non-quantum electrodynamics. Note in particular that once the statistical ensemble of metrics has been chosen, there is no gauge choice involved since the definition of apparent matter is manifestly covariant (however, the choice of the ensemble may, in some cases, reflect the point of view of some particular observer, akin to fixing a preferred referential for the observations). Another important point is that this averaging scheme is the only one which ensures that motion in the mean field can be interpreted, at least locally, as the average of "real" unaveraged motions [8, 7].

2.3 Small amplitude fluctuations

We now investigate the case when the metrics $g_{\mu\nu}(\omega)$ are all close to a reference metric ${g^{\text{ref}}}_{\mu\nu}$. More precisely, we assume that there is a small parameter ε such that, for any value of the random parameter ω,

$$g_{\mu\nu}(\omega) = {g^{\text{ref}}}_{\mu\nu} + \varepsilon {g^{(1)}}_{\mu\nu}(\omega) + \varepsilon^2 {g^{(2)}}_{\mu\nu}(\omega) + O(\varepsilon^3) \tag{4}$$

and we will expand the theory above at second order in ε.

Of course, any arbitrary choice of $g^{(1)}$ and $g^{(2)}$ will define a solution of the Einstein equation by setting the value of the stress-energy tensor to $\mathcal{T} = G/8\pi$, but these solutions are physically relevant only if the associated stress-energy tensor has a physical interpretation. In the sequel we will focus on choices of $g^{(1)}$ and $g^{(2)}$ arising from physically interesting stress-energy tensors, such as gravitational waves or fluctuations of the density of matter.

We now derive a perturbative expression for ${\mathcal{T}^{\text{app}}}^{\nu}_{\mu}$. Denote by $\mathcal{D}G$ and $\mathcal{D}^2 G$, respectively, the functional derivative and the functional Hessian of the Einstein tensor $G^{\nu}_{\mu}(g^{\text{ref}})$ with respect to variations of the metric g^{ref}. So by definition we have the expansion

$$\begin{aligned} G^{\nu}_{\mu}(g(\omega)) = G^{\nu}_{\mu}(g^{\text{ref}}) + \varepsilon(\mathcal{D}G^{\nu}_{\mu})(g^{(1)}(\omega)) + \varepsilon^2(\mathcal{D}G^{\nu}_{\mu})(g^{(2)}(\omega)) \\ + \frac{\varepsilon^2}{2}(\mathcal{D}^2 G^{\nu}_{\mu})(g^{(1)}(\omega), g^{(1)}(\omega)) + O(\varepsilon^3) \end{aligned} \tag{5}$$

which yields

$$\begin{aligned} 8\pi \left\langle T^{\nu}_{\mu}(\omega) \right\rangle = \left\langle G^{\nu}_{\mu}(g(\omega)) \right\rangle = G^{\nu}_{\mu}(g^{\text{ref}}) + \varepsilon \left\langle (\mathcal{D}G^{\nu}_{\mu})(g^{(1)}(\omega)) \right\rangle + \varepsilon^2 \left\langle (\mathcal{D}G^{\nu}_{\mu})(g^{(2)}(\omega)) \right\rangle \\ + \frac{\varepsilon^2}{2} \left\langle (\mathcal{D}^2 G^{\nu}_{\mu})(g^{(1)}(\omega), g^{(1)}(\omega)) \right\rangle + O(\varepsilon^3) \end{aligned} \tag{6}$$

It is important to note here that $\mathcal{D}G$, being a functional derivative, is by definition a linear operator in its arguments $g^{(1)}$ or $g^{(2)}$. One thus has

$$\left\langle (\mathcal{D}G^{\nu}_{\mu})(g^{(1)}(\omega)) \right\rangle = (\mathcal{D}G^{\nu}_{\mu}) \left(\left\langle g^{(1)}(\omega) \right\rangle \right) \tag{7}$$

and likewise for $g^{(2)}$. But this is not true of the Hessian $\mathcal{D}^2 G$, which is a quadratic (as opposed to linear) operator.

Meanwhile, the mean metric $\bar{g}$ is given by

$$\bar{g}_{\mu\nu} = g^{\mathrm{ref}}{}_{\mu\nu} + \varepsilon \left\langle g^{(1)}{}_{\mu\nu}(\omega) \right\rangle + \varepsilon^2 \left\langle g^{(2)}{}_{\mu\nu}(\omega) \right\rangle + O(\varepsilon^3) \tag{8}$$

so that the associated Einstein tensor is

$$\begin{aligned} \bar{G}^{\nu}_{\mu} = G^{\nu}_{\mu}(\bar{g}) = G^{\nu}_{\mu}(g^{\mathrm{ref}}) + \varepsilon(\mathcal{D}G^{\nu}_{\mu}) \left(\left\langle g^{(1)}(\omega) \right\rangle \right) + \varepsilon^2 (\mathcal{D}G^{\nu}_{\mu}) \left(\left\langle g^{(2)}(\omega) \right\rangle \right) \\ + \frac{\varepsilon^2}{2} (\mathcal{D}^2 G^{\nu}_{\mu}) \left(\left\langle g^{(1)}(\omega) \right\rangle, \left\langle g^{(1)}(\omega) \right\rangle \right) + O(\varepsilon^3) \end{aligned} \tag{9}$$

From these results, by comparing $\bar{G}^{\nu}_{\mu}$ to $\left\langle G^{\nu}_{\mu}(g(\omega)) \right\rangle$ we can directly compute the apparent stress-energy tensor:

$$\mathcal{T}^{\mathrm{app}}{}^{\nu}_{\mu} = \frac{\varepsilon^2}{16\pi} \left((\mathcal{D}^2 G^{\nu}_{\mu}) \left(\left\langle g^{(1)}(\omega) \right\rangle, \left\langle g^{(1)}(\omega) \right\rangle \right) - \left\langle (\mathcal{D}^2 G^{\nu}_{\mu})(g^{(1)}(\omega), g^{(1)}(\omega)) \right\rangle \right) + O(\varepsilon^3) \tag{10}$$

which is generally non-zero due to the quadratic nature of $\mathcal{D}^2 G$.

It is to be noted that the effect is at second order in ε, which was to be expected since at first order, gravitation is by definition linear. What is more interesting is that $g^{(2)}$ does not appear in the result. This reflects the fact that non-linearities acting on the second-order term $g^{(2)}$ will only produce higher-order terms.

It often makes sense to define the fluctuations in terms of the sources rather than the metric, i.e. to prescribe physically meaningful fluctuations $\mathcal{T}^{(1)}$ and $\mathcal{T}^{(2)}$ of the stress-energy tensor and to look for $g^{(1)}$ and $g^{(2)}$ solving the Einstein equation. That $g^{(2)}$ vanishes from the result means that, to compute the effect, it is actually enough to solve the linearized Einstein equation around g^{ref}.

A case of particular interest is when the fluctuations are "centered" i.e. when the average of the fluctuations is zero at first order:

$$\left\langle g^{(1)}(\omega) \right\rangle = 0 \tag{11}$$

in which case we simply get

$$\mathcal{T}^{\mathrm{app}}{}^{\nu}_{\mu} = -\frac{\varepsilon^2}{16\pi} \left\langle (\mathcal{D}^2 G^{\nu}_{\mu})(g^{(1)}(\omega), g^{(1)}(\omega)) \right\rangle \tag{12}$$

at this order in ε.

3 Fluctuations around dust cosmologies

3.1 Basics

We now apply the above to the case of either gravitational waves or density fluctuations around a homogeneous and isotropic, spatially flat dust universe (flat Friedmann–Lemaître–Robertson–Walker metric). The reference metric and stress-energy tensor of such a space-time are, in conformal coordinates [21]:

$$g^{\text{ref}} = a(\eta)^2(d\eta^2 - dx^2 - dy^2 - dz^2) \qquad T^0_0 = \rho(\eta) \quad T^0_i = T^j_i = 0 \tag{13}$$

where a is the so-called expansion factor and ρ is the energy density. The Einstein equation delivers $a(\eta) = C\eta^2$ and $8\pi\rho(\eta) = 3\dot{a}^2/a^4 = 12/C^2\eta^6$, where C is an arbitrary (positive) constant. Proper time is $\tau = C\eta^3/3$ and the Hubble "constant" is $H = \dfrac{1}{a}\dfrac{da}{d\tau} = \dfrac{\dot{a}}{a^2} = \dfrac{2C}{\eta^3}$.

The perturbations $g^{(1)}$ considered in this article will be of two types: gravitational waves and matter density fluctuations. They can be written as sums or integrals of spatial Fourier modes (this makes sense since g^{ref} is spatially flat). Each term in such a series is of the form $F(\eta)\exp\left(i(\mathbf{q}.\mathbf{r} + \omega_{\mathbf{q}})\right)$ where $F(\eta)$ is some tensor, $\mathbf{q}$ is a three-dimensional wave vector, and $\omega_{\mathbf{q}}$ is a phase associated with mode $\mathbf{q}$. Averaging a given mode $\mathbf{q}$ on spatial scales much larger than the wave-length $1/\left|\mathbf{q}\right|$ is equivalent to averaging this mode over the phase $\omega_{\mathbf{q}} \in [0; 2\pi]$. We therefore choose the set of all phases $(\omega_{\mathbf{q}})$ as our random parameter, and perform all averagings over these phases. By a simple superposition argument, which we omit, one can easily check that if several Fourier modes are present but statistically independent, then the averaging can be performed separately for each mode (at least at second order). Hence, in the sequel, we will use a single Fourier mode.

3.2 Gravitational waves

Consider a single gravitational wave propagating along the above background [22]. This wave admits two polarizations [21]; since the background is isotropic, there is no loss of generality in assuming the wave propagates along, say the x-axis. The first-order metric perturbation then reads, for the first polarization:

$$g^{(1)}{}_{22}(\omega) = -a(\eta)^2 e^{i(q(x-\eta)+\omega)}(1 - i/q\eta)/\eta^2 \qquad g^{(1)}{}_{33} = -g^{(1)}{}_{22} \tag{14}$$

with the other components equal to 0. Here q is the wave number in conformal coordinates, and $\omega \in [0; 2\pi]$.

The statistical averaging corresponds to a uniform averaging over $\omega \in [0; 2\pi]$; this models situations in which the system is observed at a resolution much larger than the perturbation wavelength $1/q$.

The quantity $n_{\text{osc}} = q\eta$ measures the typical number of oscillations (periods) in that part of the universe accessible to an observer situated at time η. The relative amplitude of the perturbation $\varepsilon g^{(1)}$ at time η, compared to g^{ref}, is $\tilde{\varepsilon}(\eta) = \varepsilon/\eta^2$. We will express the results in terms of those quantities.

The stress-energy tensor of apparent matter is then given by (12). In practice the Hessian term $\mathcal{D}^2 G^\nu_\mu$ in (12) is readily obtained as the ε^2 term in a Taylor expansion of the Einstein tensor of the metric $g^{\text{ref}} + \varepsilon g^{(1)}$, which can be computed using any symbolic computation software.

Using the real part of the metric above, i.e. $-a(\eta)^2(\cos(q(x-\eta)+\omega) + \frac{1}{q\eta}\sin(q(x-\eta)+\omega))/\eta^2$, we get, at second order in ε :

$$\mathcal{T}^{\text{app}}{}^0_0 = \tilde{\varepsilon}(\eta)^2 n_{\text{osc}}^2 \, \frac{1 - 14/n_{\text{osc}}^2 - 39/2n_{\text{osc}}^4}{48} \, \rho(\eta) \tag{15}$$

$$\mathcal{T}^{\text{app}}{}^1_1 = -\tilde{\varepsilon}(\eta)^2 n_{\text{osc}}^2 \, \frac{1 - 2/n_{\text{osc}}^2 - 27/2n_{\text{osc}}^4}{48} \, \rho(\eta) \tag{16}$$

$$\mathcal{T}^{\text{app}}{}^2_2 = \mathcal{T}^{\text{app}}{}^3_3 = \tilde{\varepsilon}(\eta)^2 n_{\text{osc}}^2 \, \frac{1/n_{\text{osc}}^2 + 9/2n_{\text{osc}}^4}{48} \, \rho(\eta) \tag{17}$$

$$\mathcal{T}^{\text{app}}{}^1_0 = \tilde{\varepsilon}(\eta)^2 n_{\text{osc}}^2 \, \frac{1}{48} \, \rho(\eta). \tag{18}$$

All other components are 0 and all these relations hold up to $O(\varepsilon^3 n_{\text{osc}}^2 + \varepsilon^2)$.

If instead of a single wave, we consider a superposition of statistically independent gravitational waves sharing a common frequency and amplitude, but propagating along random spatial directions, we get a spatially isotropic version of (15–18), namely

$$\mathcal{T}^{\text{app}}{}^0_0 = \tilde{\varepsilon}(\eta)^2 n_{\text{osc}}^2 \, \frac{1 - 14/n_{\text{osc}}^2 - 39/2n_{\text{osc}}^4}{48} \, \rho(\eta) \tag{19}$$

$$\mathcal{T}^{\text{app}}{}^1_1 = \mathcal{T}^{\text{app}}{}^2_2 = \mathcal{T}^{\text{app}}{}^3_3 = -\tilde{\varepsilon}(\eta)^2 n_{\text{osc}}^2 \, \frac{1/3 - 4/3n_{\text{osc}}^2 - 45/6n_{\text{osc}}^4}{48} \, \rho(\eta) \tag{20}$$

These expressions show that a background of gravitational waves of high frequency ($n_{\text{osc}} \gg 1$) behaves like an ordinary stress-energy tensor for radiation, with positive pressure equal to a third of its energy density (at this order in $\tilde{\varepsilon}(\eta)^2 n_{\text{osc}}^2$).

The first-order metric perturbation for the other polarization is given by

$$g^{(1)}{}_{23}(\omega) = (C^2\eta^4) e^{i(q(x-\eta)+\omega)}(1 - i/q\eta)/\eta^2 \tag{21}$$

The stress-energy tensor of the apparent matter associated with this polarization is identical to (19) and (20) and does not warrant separate discussion.

Orders of magnitude. The important factor in (19) and (20) is $\tilde{\varepsilon}(\eta)^2 n_{\text{osc}}^2$. The energy density and pressure of apparent matter are (at this order) quadratic, not only in the amplitude $\tilde{\varepsilon}(\eta)$ of the perturbation, but also in its "frequency" n_{osc}. Thus, $\tilde{\varepsilon}(\eta) \ll 1$ does not necessarily translate into negligible energy density and pressure of apparent matter: the smallness of $\tilde{\varepsilon}(\eta)$ can be compensated by a sufficiently high frequency n_{osc}. For example, gravitational waves of relative amplitude $\tilde{\varepsilon}(\eta) \approx 10^{-5}$ and oscillation number $n_{\text{osc}} \approx 10^5$ would generate an effective apparent large-scale stress-energy comparable to the energy density of the dust present in this model. Note that such a wave would have today a physical frequency of order 10^{-12} Hz (corresponding to a wavelength of $1/n_{\rm osc}$ times the radius of the observable Universe). The presence of a wave with such characteristics would not be contradicted by current evidence: it would elude direct observation [23] and, as far as indirect constraints are concerned, the best upper bounds [24, 25, 26, 27, 28] around this frequency are $\Omega_{\rm GW}\, h_{100}^2 \leq 0.1$ and $\Omega_{\rm GW}\, h_{100}^2 \leq 0.5$, which are quite compatible with a $\mathcal{T}^{\rm app}$ of order unity.

This example is merely an illustration of the possible order of magnitude of the apparent matter effects from a theoretical viewpoint: in particular, we do not claim that gravitational waves with these exact frequency and amplitude actually exist in the Universe (though, given the wavelike nature of the Einstein equation, from a purely mathematical point of view it is the absence of gravitational waves, rather than their presence, that needs a justification). But, at least, this example clearly shows that the effect of such fluctuations on the Universe must definitely not be neglected unless really compelling arguments rule them out.

3.3 Fluctuations in the density of matter

The first-order expressions for the metric and stress-energy tensors corresponding to a matter density fluctuation around a spatially flat, homogeneous and isotropic universe are well-known and given in [21]. These expressions can be used to compute the stress-energy of apparent matter from the formula (12).

We discuss here the simplest such perturbation; other types of density fluctuations are presented in the Appendix. The perturbation is of the form

$$\begin{aligned} &{g^{(1)}}_{00} = 0 \qquad {g^{(1)}}_{11}(\omega) = a(\eta)^2\eta^2\cos(qx+\omega) \\ &{g^{(1)}}_{22}(\omega) = {g^{(1)}}_{33}(\omega) = -a(\eta)^2\frac{10}{q^2}\cos(qx+\omega), \end{aligned} \tag{22}$$

corresponding to the following first-order stress-energy tensor perturbation

$$\mathcal{T}^{(1)}{}^0_0(\omega) = \rho\,\frac{\eta^2}{2}\cos(qx+\omega) \qquad \mathcal{T}^{(1)}{}^j_i = \mathcal{T}^{(1)}{}^j_0 = 0. \tag{23}$$

This stress-energy tensor describes a co-moving, shear-free spatial density fluctuation. Note that the relative amplitude of the perturbation increases with time, which traces the aggregating effect of gravitation.

The quantity $\tilde{\varepsilon}(\eta) = \varepsilon\eta^2$ measures the effective relative magnitude of the perturbation $\varepsilon g^{(1)}$ with respect to $g^{\rm ref}$. The quantity $n_{\rm osc} = q\eta$ represents the number of oscillations (periods) in that part of the universe accessible to an observer situated at time η; typically $n_{\rm osc} \gg 1$.

As above, the averaging is over $\omega \in [0; 2\pi]$, and the stress-energy tensor of apparent matter can be obtained from (12) by direct computation. This gives

$$\mathcal{T}^{\mathrm{app}\,0}_{\;\;0} = -\tilde{\varepsilon}(\eta)^2 \frac{1 - 75/n_{\mathrm{osc}}^2}{16\pi} \rho(\eta) \tag{24}$$

$$\mathcal{T}^{\mathrm{app}\,1}_{\;\;1} = \tilde{\varepsilon}(\eta)^2 \frac{25}{16\pi n_{\mathrm{osc}}^2} \rho(\eta) \tag{25}$$

$$\mathcal{T}^{\mathrm{app}\,2}_{\;\;2} = \mathcal{T}^{\mathrm{app}\,3}_{\;\;3} = \tilde{\varepsilon}(\eta)^2 \frac{7 + 50/n_{\mathrm{osc}}^2}{32\pi} \rho(\eta) \tag{26}$$

and all the other terms are 0 at this order in ε.

The apparent matter associated with these fluctuations is thus characterized, at this order, by a negative energy density and a negative pressure. Loosely speaking, the negative energy could be interpreted in a semi-Newtonian setting as the gravitational energy of the fluctuations and the negative pressure represents the collapsing effects of gravitation.

There is an important difference with respect to the gravitational wave case above, namely that the effect simply scales like the square of the effective amplitude of the perturbation, with no n_{osc}^2 factor (compare with (19–20)). Thus, the net large-scale effect of high-frequency gravitational waves is much more important than the net large-scale effect of matter density fluctuations of comparable wavelength, at least at this order in ε.

4 Conclusion

We have investigated how small-scale fluctuations influence the homogeneous and isotropic large-scale expansion of cosmological models. We have restricted the discussion to dust models and studied fluctuations in matter density as well as gravitational waves. Our perturbative results indicate that the so-called back reaction effect is dominated by gravitational waves, rather than matter density fluctuations. The relative importance of the effective large-scale stress-energy generated by gravitational waves scales as the squared product of their amplitude by their frequency. Thus, even small amplitude waves can generate an important effect provided their frequencies are high enough. For example, it is found that waves of current amplitude $\sim 10^{-5}$ and current physical frequency 10^{-12} Hz, which are not detectable with today's technology, would generate a large-scale stress-energy comparable to the dust energy.

The equation of state of the large-scale stress-energy generated by an isotropic background of gravitational waves is simply the equation of state of radiation with postive energy density and pressure. On the other hand, the matter density fluctuations we studied lead to negative energy density and pressure.

The results presented here prove that small-scale fluctuations can influence drastically the large-scale expansion of the universe and that back reaction cannot be *a priori* neglected in cosmology. One can then wonder if at least part of the cosmological dark energy cannot be interpreted as a large-scale signature of such small-scale fluctuations. The material presented in this article is not yet sufficient to reach a definitive conclusion in this matter. Let us nevertheless remark that the extremely simple cosmological models considered in this manuscript are already rich enough to generate apparent matters with very different equations of state, and that equations of state strongly ressembling that of the cosmological dark energy has been found by averaging a Schwarzschild black hole [18]. This work thus needs to be extended in several directions before a clear-cut conclusion can be reached. First, computations should be carried out on more general models than flat dust cosmologies. Second, the non-perturbative regime should be addressed, for example by numerical simulations. Third, different types of fluctuations should be combined and allowed to interfere with each other.

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Appendix: More on dust density fluctuations

First-order perturbations of a Friedmann–Lemaître–Robertson–Walker metric are described in [21] and are of various types. One of them is the gravitational wave considered in Section 3.2. For dust cosmologies, the next one reads:

$$\begin{gathered} g^{(1)}{}_{00} = 0 \qquad g^{(1)}{}_{11} = -a(\eta)^2\beta(\eta)\cos(qx+\omega) \\ g^{(1)}{}_{22} = g^{(1)}{}_{33} = -a(\eta)^2\gamma(\eta)\cos(qx+\omega)/q^2; \end{gathered} \tag{27}$$

associated with first-order fluctuations of matter

$$\mathcal{T}^{(1)}{}^0_0 = \frac{\cos(qx+\omega)}{8\pi a^3}\left(a\gamma + 2\dot{a}\dot{\gamma}/q^2 + \dot{a}\dot{\beta}\right) \quad \mathcal{T}^{(1)}{}^1_0 = \frac{\dot{\gamma}\sin(qx+\omega)}{8\pi q a^2}, \tag{28}$$

the other components being 0. Here according to [21], the functions $\beta(\eta)$ and $\gamma(\eta)$ must satisfy

$$\ddot{\gamma} + 2\frac{\dot{a}}{a}\dot{\gamma} = 0 \qquad \ddot{\beta} + 2\frac{\dot{a}}{a}\dot{\beta} + \gamma = 0 \tag{29}$$

(The case given in the text is the simplest solution $\beta = -\eta^2, \gamma = 10$.)

Our expression (12) for apparent matter yields

$$\mathcal{T}^{\mathrm{app}}{}^0_0 = \frac{\varepsilon^2}{64\pi a^2}\left(3\gamma^2/q^2 + 2\beta\gamma - \dot{\gamma}^2/q^4 - 2\dot{\beta}\dot{\gamma}/q^2 + 4\frac{\dot{a}}{a}\beta\dot{\beta} + 8\frac{\dot{a}}{a}\gamma\dot{\gamma}/q^4\right) \tag{30}$$

$$\mathcal{T}^{\mathrm{app}}{}^1_1 = \frac{\varepsilon^2}{64\pi a^2}\left(\gamma^2/q^2 + \dot{\gamma}^2/q^4\right) \tag{31}$$

$$\mathcal{T}^{\mathrm{app}}{}^2_2 = \mathcal{T}^{\mathrm{app}}{}^3_3 = \frac{\varepsilon^2}{64\pi a^2}\left(\gamma^2/q^2 - \beta\gamma + \dot{\beta}^2 + \dot{\gamma}^2/q^4 - \dot{\beta}\dot{\gamma}/q^2\right) \tag{32}$$

and the other components are 0 or $O(\varepsilon^3)$.

In the regime we are interested in, $q \gg 1$, this reduces to

$$\mathcal{T}^{\mathrm{app}}{}^0_0 = \frac{\varepsilon^2}{32\pi a^2}\left(\beta\gamma + 2\frac{\dot{a}}{a}\beta\dot{\beta}\right) \tag{33}$$

$$\mathcal{T}^{\mathrm{app}}{}^1_1 = 0 \tag{34}$$

$$\mathcal{T}^{\mathrm{app}}{}^2_2 = \mathcal{T}^{\mathrm{app}}{}^3_3 = \frac{\varepsilon^2}{64\pi a^2}\left(\dot{\beta}^2 - \beta\gamma\right) \tag{35}$$

Since β and γ satisfy the second-order differential system (29), we can prescribe β, $\dot{\beta}$, γ and $\dot{\gamma}$ arbitrarily at one point in time. In particular, this leads to arbitrary signs for the energy and pressure of apparent matter. However, for large η, the system (29) implies that γ will tend to a constant and β will grow in time like η^2: this is the most interesting case, discussed in Section 3.3.

The last type of perturbation mentioned in [21] corresponds to a pure sheer perturbation; it takes the form

$$g^{(1)}{}_{12} = g^{(1)}{}_{21} = -a(\eta)^2\beta(\eta)\cos(qx+\omega)/q \tag{36}$$

corresponding to first-order stress-energy perturbation

$$\mathcal{T}^{(1)}{}^2_0 = -\mathcal{T}^{(1)}{}^0_2 = \frac{\dot{\beta}\sin(qx)}{2a^2} \tag{37}$$

where β satisfies $\ddot{\beta} + 2\frac{\dot{a}}{a}\dot{\beta} = 0$ i.e. $\beta = C_1/\eta^3$ in our case (the integration constant is a gauge choice).

For apparent matter this yields

$$\mathcal{T}^{\mathrm{app}}{}^0_0 = \frac{\varepsilon^2}{64\pi a^2 q^2}\left(\dot{\beta}^2 + 8\frac{\dot{a}}{a}\beta\dot{\beta}\right) \tag{38}$$

$$\mathcal{T}^{\mathrm{app}}{}^1_1 = \mathcal{T}^{\mathrm{app}}{}^2_2 = \frac{\varepsilon^2\dot{\beta}^2}{64\pi a^2 q^2} \qquad \mathcal{T}^{\mathrm{app}}{}^3_3 = \frac{3\varepsilon^2\dot{\beta}^2}{64\pi a^2 q^2} \tag{39}$$

So, not only do sheer perturbations decrease with time like $1/\eta^3$, but the apparent matter effect is small at high frequencies.

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