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Large scale nonlinear effects of fluctuations in relativistic gravitation

C. Chevalier [a,b,*], F. Debbasch [b], Y. Ollivier [c]

[a] Institute for Theoretical Physics, University of Amsterdam, Valckenierstraat 65 1018 XE Amsterdam, The Netherlands

[b] Université Pierre et Marie Curie-Paris 6, UMR 8112, ERGA-LERMA, 3 rue Galilée 94200 Ivry, France

[c] CNRS, UMPA, Ecole normale supérieure de Lyon, F-69364 Lyon Cedex 07, France

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ABSTRACT

The first fully nonlinear mean field theory of relativistic gravitation was developed in 2004. The theory makes the striking prediction that averaging or coarse graining a gravitational field changes the apparent matter content of space–time. A review of the general theory is presented, together with applications to black hole and cosmological space–times. The results strongly suggest that at least part of the dark energy may be the net large scale effect of small scale fluctuations around a mean homogeneous isotropic cosmology.

1. Introduction

General relativity is a nonlinear theory and, hence, small scale phenomena may have a non-trivial average effect at large scales. Since, at the same time, astrophysical and cosmological observations have only finite space and time resolutions, it is in practice necessary [1] to develop a mean field theory of gravitation, i.e. an effective theory allowing a self-consistent description of the observed gravitational field at a given scale or resolution, accounting for the average net effects of small scale phenomena not accessible within a given observational setup.

Developing such an effective theory has long been a subject of active research [2–8]. The first general mean field theory for Einstein gravitation was obtained four years ago [9,10]. The theory makes the striking prediction that averaging or coarse graining a gravitational field changes the apparent matter content of space–time. In particular, the net 'large scale' effect of the averaged-upon, 'small scale' gravitational degrees of freedom is to contribute an 'apparent matter' at large scale, necessary to account for the coarse grained gravitational field. This matter may be charged if the gravitational field is coupled to an electromagnetic field.

This contribution is organized as follows. We first introduce the general mean field theory. Then we address perturbatively the important example of background gravitational waves around a simple homogeneous and isotropic, spatially flat dust universe; our results show, at least for this very simple model, that there is a frequency and amplitude range in which background waves, while being undetectable with current techniques, would generate an apparent large scale matter of energy density comparable to the energy density of the dust.

Finally we present coarse grainings of both the Schwarzschild and the extreme Reisner–Nordström (RN) black holes. In particular, the Schwarzschild black hole, which is a vacuum solution of the Einstein field equations, is shown to appear, after coarse graining, as surrounded by an apparent matter whose equation of state strongly resembles the equation of state commonly postulated for cosmological dark energy. We also investigate thermodynamical aspects, highlighting the fact that the envisaged coarse graining transforms the extreme RN black hole, which has a vanishing temperature, into a black hole of non-vanishing temperature.

* Corresponding author at: Université Pierre et Marie Curie-Paris 6, UMR 8112, ERGA-LERMA, 3 rue Galilée 94200 Ivry, France.
E-mail address: mailto:chevalier_claire@yahoo.frchevalier_claire@yahoo.fr (C. Chevalier).

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2. A mean field theory for general relativity

Let $\mathcal{M}$ be a fixed manifold and let Ω be an arbitrary probability space. Let $g(\omega)$ be an ω-dependent Lorentzian metric defined on $\mathcal{M}$; let also $A(\omega)$ be an ω-dependent electromagnetic 4-potential, with associated current $j(\omega)$. Each triplet $\mathcal{S}(\omega) = (\mathcal{M}, g(\omega), A(\omega))$ represents a physical space–time depending on the random parameter $\omega \in \Omega$. For example, $g(\omega)$ may represent a gravitational wave of random phase and wavevector around a given reference space–time.

With each space–time $\mathcal{S}(\omega)$ are associated the Einstein tensor $E(\omega)$ of the metric $g(\omega)$, and a stress–energy tensor $\mathcal{T}(\omega)$ satisfying the Einstein equation

$$E(\omega) = 8\pi \mathcal{T}(\omega). \tag{1}$$

We write

$$\mathcal{T}(\omega) = \mathcal{T}_{(A(\omega),g(\omega))} + \mathcal{T}_m(\omega), \tag{2}$$

where $\mathcal{T}_{(A(\omega),g(\omega))}$ represents the electromagnetic stress–energy tensor generated by $A(\omega)$ in $g(\omega)$, and $\mathcal{T}_m(\omega)$ represents the stress–energy tensor of other matter fields.

It has been shown in [9] that such a collection of space–times can be used to define a single, mean or coarse grained space–time $(\mathcal{M}, \bar{g}, \bar{A})$ representing the average, "macroscopic" behavior of these random space–times. The metric $\bar{g}$ and the potential $\bar{A}$ are the respective averages of the metrics $g(\omega)$ and of the potentials $A(\omega)$ over ω; thus, for all points P of $\mathcal{M}$,

$$\bar{g}(P) = \langle g(P, \omega) \rangle \tag{3}$$

and

$$\bar{A}(P) = \langle A(P, \omega) \rangle, \tag{4}$$

where the brackets on the right-hand side indicate an average over the probability space Ω. If we think of $g(\omega)$ as a reference metric perturbed by small random contributions, then $\bar{g}$ will represent the average metric, where the fluctuations have been smoothed out but with the same macroscopic behavior.

The metric $\bar{g}$ defines an Einstein tensor $\bar{\mathcal{E}}$ for the coarse grained space–time. However, since the expression for the Einstein tensor as a function of the metric is nonlinear, this Einstein tensor and the average stress–energy tensor will, in general, *not* be related by the Einstein equation:

$$\bar{\mathcal{E}} \neq 8\pi \langle \mathcal{T}(\omega) \rangle$$

so physical measurements attempting to relate the coarse grained space–time to its average matter content would yield a violation of the Einstein equation. This would not happen in a Newtonian setting since, then, the relation between field and matter is linear and thus is unchanged under averaging.

To enforce validity of the Einstein equation for the coarse grained space–time, it is thus necessary to introduce a new term:

$$T^{\mathrm{app}\,\alpha}{}_{\beta} = \bar{\mathcal{E}}^{\alpha}_{\beta}/8\pi - \langle (\mathcal{T}_m)^{\alpha}_{\beta}(\omega) \rangle - \mathcal{T}_{(\bar{A},\bar{g})}{}^{\alpha}{}_{\beta}, \tag{5}$$

so that the stress–energy tensor of the coarse grained space–time can be described as the sum of the stress–energy tensor of the average quadripotential $\bar{A}$, of the average stress–energy tensor $\langle (\mathcal{T}_m)(\omega) \rangle$ appearing in the averaged space–times, and of this new term T^{app}. This generally non-vanishing tensor field can be interpreted as the stress–energy tensor of an 'apparent matter' in the coarse grained space–time. This apparent matter describes the cumulative nonlinear effects of the averaged-out small scale fluctuations of the gravitational and electromagnetic fields on the large scale behavior of the coarse grained gravitational field.

In particular, even the vanishing of $\mathcal{T}(\omega)$ for all ω does not necessarily imply the vanishing of $\bar{\mathcal{T}}$. The mean stress–energy tensor $\bar{\mathcal{T}}$ can therefore be non-vanishing in regions where the unaveraged stress–energy tensor actually vanishes.

The Maxwell equation relating the electromagnetic potential to the electromagnetic current also couples the electromagnetic field and the gravitational field non-linearly; the mean current $\bar{j}$ associated with $\bar{A}$ in $\bar{g}$ does not therefore coincide with the average $\langle j(\omega) \rangle$. In particular, a region of space–time where $j(\omega)$ vanishes for all ω is generally endowed with a non-vanishing mean current $\bar{j}$.

Let us finally mention that the averaging scheme just presented is the only one which ensures that motion in the mean field can actually be interpreted, at least locally, as the average of 'real' unaveraged motions. This important point is fully developed in [10].

3. Waves around a homogeneous isotropic simple cosmology

The averaging procedure above has been applied to background gravitational waves [11] propagating around a homogeneous isotropic, spatially flat dust universe. The main conclusion is that the large scale effect of these gravitational waves is close to that of a matter field with positive energy and pressure, whose order of magnitude is roughly $n^2\varepsilon^2$ where n is the relative frequency of the waves and ε their relative amplitude. In particular, in some regimes this energy would be comparable to that of the dust, even for some currently undetectable gravitational waves.

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The reference metric and stress–energy tensor are those of the flat Friedmann–Lemaître–Robertson–Walker (FLRW) universe, which, in conformal coordinates, read

$$g^{\mathrm{ref}} = a(\eta)^2(\mathrm{d}\eta^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2) \qquad T_0^0 = \rho(\eta) \quad T_i^0 = T_i^j = 0 \tag{6}$$

where a is the so-called expansion factor and ρ is the energy density. The Einstein equation delivers $a(\eta) = C\eta^2$ and $8\pi\rho(\eta) = 3\dot{a}^2/a^4 = 12/C^2\eta^6$, with C an arbitrary (positive) constant. The proper time is $\tau = C\eta^3/3$ and the Hubble 'constant' is $H = \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}\tau} = \frac{\dot{a}}{a^2} = \frac{2C}{\eta^3}$.

We will assume that the averaging scale is much larger than the wavelength of the gravitational waves; this means that we can represent the waves as a statistical ensemble of Fourier series with random phase $\omega \in [0; 2\pi]$.

So let us consider a gravitational wave propagating around the homogeneous and isotropic FLRW background. By isotropy we can assume that the wave propagates in the direction x. Such a gravitational wave is represented at first order by the metric perturbation

$$g^{(1)}{}_{22} = -\varepsilon(\eta)a(\eta)^2 \mathrm{e}^{\mathrm{i}q(x-\eta)}(1 - i/q\eta) \qquad g^{(1)}{}_{33} = -g^{(1)}{}_{22} \tag{7}$$

for the first polarization (the other polarization yields identical results and thus will not be discussed). Here the constant q is the wavenumber in conformal coordinates, and the relative amplitude of the wave is given by $\varepsilon(\eta) = \varepsilon_0/\eta^2$ for some constant ε_0. The number of oscillations (periods) in that part of the universe accessible to an observer situated at time η is $n_{\mathrm{osc}} = q\eta$.

Using the real part of the above, i.e. $-\varepsilon(\eta)\,a(\eta)^2(\cos(q(x-\eta)) + \frac{1}{q\eta}\sin(q(x-\eta)))$, we can compute the apparent matter associated with this gravitational wave and compare it to the energy density $\rho(\eta)$ of the dust. Using (5) we get, at second order in ε and for large n_{osc},

$$T^{\mathrm{app}}{}^0_0 = \varepsilon(\eta)^2 n_{\mathrm{osc}}^2 \frac{1}{48}\rho(\eta) \tag{8}$$

$$T^{\mathrm{app}}{}^1_1 = -\varepsilon(\eta)^2 n_{\mathrm{osc}}^2 \frac{1}{48}\rho(\eta) \tag{9}$$

$$T^{\mathrm{app}}{}^1_0 = \varepsilon(\eta)^2 n_{\mathrm{osc}}^2 \frac{1}{48}\rho(\eta). \tag{10}$$

All other components are 0 and all of these relations hold up to $O(\varepsilon^3 n_{\mathrm{osc}}^2 + \varepsilon^2)$.

Consider now a superposition of statistically independent gravitational waves of type (7). Suppose that these waves share a common frequency and amplitude, but propagate along spatial directions which are distributed uniformly over the unit sphere. The stress–energy tensor associated with the superposition of these waves can be easily deduced from the above and its non-vanishing components read

$$T^{\mathrm{app}}{}^0_0 = \varepsilon(\eta)^2 n_{\mathrm{osc}}^2 \frac{1}{48}\rho(\eta) \tag{11}$$

$$T^{\mathrm{app}}{}^1_1 = T^{\mathrm{app}}{}^2_2 = T^{\mathrm{app}}{}^3_3 = -\varepsilon(\eta)^2 n_{\mathrm{osc}}^2 \frac{1}{144}\rho(\eta) \tag{12}$$

with all other components zero, up to the same order as before. The above expressions show that a background of high frequencies (i.e. $n_{\mathrm{osc}} \gg 1$) gravitational waves behaves like radiation with positive pressure equal to a third of its energy density. The energy density is (at this order) quadratic in the frequency and amplitude.

Thus, a background wave of relative amplitude $\varepsilon \approx 10^{-5}$ and oscillation number $n_{\mathrm{osc}} \approx 10^5$ would generate an effective large scale stress–energy in the universe comparable to the energy density of dust present in this model. Such a wave would correspond today to a physical frequency of order 10^{-12} Hz and would elude direct observation [12].

4. Coarse graining of black hole space–times

Both the Schwarzschild and the extreme Reisner–Nordström space–times of total mass M have been coarse grained using the above procedure [13–16]. For the Schwarzschild (resp. extreme RN) black hole, the metric $g(\omega)$ is the Schwarzschild (resp. extreme RN) metric spatially translated by ω (resp. $\mathrm{i}\omega$) in spatial Kerr–Schild coordinates [17], with ω distributed uniformly in the 3-ball $\mathcal{B}_a$ of radius $a > 0$. In both cases, exact expressions have been found for the mean metric $\bar{g}$ for all points with radial Kerr–Schild coordinate r greater than the coarse graining parameter a.

Both averaged space–times describe black holes with the following properties. The total mass of the space–times, as well as the total charge of the extreme black hole, are preserved by the averaging. But energy and mass are spatially redistributed: in particular the averaged Schwarzschild black hole is surrounded by an apparent matter with energy density ε equal to the opposite of the radial pressure p_r and the scalar curvature induced by this apparent matter is strictly negative. The similarities with dark energy are striking.

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The temperatures of the black holes are also modified by the averaging. Quite remarkably, the extreme black hole of vanishing temperature is modified into a finite temperature black hole. Indeed, the temperature of the black hole obtained by averaging the extreme RN black hole reads, at first order in the coarse graining parameter a,

$$\Theta(a, M) \simeq \frac{a}{2\sqrt{5}\pi\, M^2}. \tag{13}$$

Thus, at least some classical black holes of finite temperature can be understood as statistical superpositions of other *purely classical* (as opposed to quantum) gravitational fields.

5. Conclusion

We have reviewed the new mean field theory of relativistic gravitation and discussed some applications to black hole physics and cosmology. There are two main conclusions. The first one concerns black hole thermodynamics. We have proved that it is possible to build at least some finite temperature black holes as statistical ensembles of *classical* vanishing temperature extreme black holes. This result is striking because black hole thermodynamics has until now been understood only by building black holes as statistical ensembles of quantum objects.

The second conclusion concerns cosmology. We have proved that small scale background gravitational waves propagating around an homogeneous and isotropic universe can contribute significantly to the large scale energy density. For instance, waves with a present relative amplitude of approximately 10^{-5} and a dimensionless comoving wavenumber (oscillation number) of 10^5 would elude current observation and would be sufficient to close the universe.

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