Discrete Ricci curvature: Open problems

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Abstract

This document lists some open problems related to the notion of discrete Ricci curvature defined in [Oll09, Oll07]. Do not hesitate to contact me for precisions.

Please inform me if you seriously work on one of these problems, so that I don't put a student on it!

The problems are not ordered.

Problem A (Log-concave measures).
Ricci curvature is positive for $\mathbb{R}^N$ equipped with a Gaussian measure, and this generalizes to smooth, uniformly strictly log-concave measures. What happens for a general log-concave measure? The next example would be a convex set (whose boundary has "positive curvature" in an intuitive geometric sense), with associated process a Brownian motion conditioned not to leave the convex body.

Problem B (Finsler manifolds).
Ricci curvature is 0 for $\mathbb{R}^N$ equipped with an L^p norm. Does this give anything interesting in Finsler manifolds? (Compare [Oht] and forthcoming work by Ohta and Sturm using the displacement convexity definition.)

Problem C (Nilpotent groups).
Ricci curvature of $\mathbb{Z}^N$ is 0. What happens on discrete or continuous nilpotent groups?

For example, on the discrete Heisenberg group $\langle\, a, b, c \mid ac = ca, bc = cb, [a, b] = c \,\rangle$, the natural discrete random walk analogous to the hypoelliptic diffusion operator on the continuous Heisenberg group is the random walk generated by a and b. Since these generators are free up to length 8, clearly Ricci curvature is negative at small scales, but does it tend to 0 at larger and larger scales?

Problem D (Continuous-time).
In lots of examples, the natural process is a continuous-time one. When the space is finite or compact, or when one has good explicit knowledge of the process (as for Ornstein–Uhlenbeck on $\mathbb{R}^N$), discretization works very well, but this might not be the case in full generality.

Suppose a continuous-time Markov semigroup $(m_x^t)_{x\in X, t\in\mathbb{R}_+}$ is given. One can define a Ricci curvature in a straightforward manner as

$$\kappa(x, y) := \liminf_{t\to 0^+} \frac{1}{t} \frac{d(x, y) - \mathcal{T}_1(m_x^t, m_y^t)}{d(x, y)}$$

but then, in the proofs of the elementary properties above, there arise non-trivial issues of commutation between limits and integrals, especially if the generator of the process is unbounded. Is the definition above, combined with some assumption on the process (e.g. non-explosion), enough to get all the properties above in full generality, for both diffusions and jump processes? Given an unbounded generator for the process, is positivity of the $\kappa(x, y)$ above enough to ensure non-explosion? (One could directly use the Lipschitz norm contraction as a definition [RS05, Jou07], but first, this is not a local criterion, and second, it only defines a lower bound on Ricci curvature, not a value at a given point.) (These questions are currently being solved by my student Laurent Veysseire.)

Problem E (Non-reversible spectral gap).
Ricci curvature gives a spectral gap bound when the random walk is reversible or when the space is finite. What happens to the spectral gap in the non-reversible case? There are different ways to formulate the question: spectral radius of the averaging operator, operator norm, Poincaré inequality. (Note that a Poincaré inequality with a worse constant and with a "blurred" gradient always holds, cf. the section on log-Sobolev inequality in [Oll09].) Is is possible to use a finite-space approximation?

Problem F (Sharp Lichnerowicz theorem).
For the ε-step random walk on a Riemannian manifold, the operator $\Delta = \mathrm{M} - \mathrm{Id}$ of the random walk behaves $\frac{\varepsilon^2}{2(N+2)}$ times the Laplace-Beltrami operator and the coarse Ricci curvature is $\frac{\varepsilon^2}{2(N+2)}$ times ordinary Ricci curvature, so that we get a spectral gap estimate of $\inf \mathrm{Ric}(v)$ for the Laplace-Beltrami operator. On the other hand, the Lichnerowicz theorem has a qualitatively comparable but slightly better spectral gap estimate $\frac{N}{N-1} \inf \mathrm{Ric}(v)$, which is sharp for the sphere. This is because our definition of $\kappa(x, y)$ somehow overlooks that the sectional curvature $K(v, v)$ in the direction of xy is 0. Is there a way to take this into account, e.g. using couplings by reflection? (Note that our estimate is sharp for the discrete cube as well as for the Ornstein–Uhlenbeck process, so the phenomenon is rather specific to the Riemannian, drift-free situation.)

Problem G (Non-constant curvature).
The estimate above uses the infimum of $\kappa(x, y)$. Is it possible to relax this assumption and, for example, include situations where κ takes "not too many" negative or zero values? Using the Ricci curvature of the iterates m_x^t for some $t \geqslant 2$ should "smoothen out" exceptional values of $\kappa(x, y)$, so that for large t the Ricci curvature of m_x^t should be close to an "average" Ricci curvature of m_x (probably involving large deviations of the average value of $\kappa(x, y)$ along trajectories of the random walk).

This may be interesting e.g. on random objects (graphs...) where locally some negative curvature is bound to occur somewhere.

Problem H (Isoperimetric profile and curvature at infinity).
Suppose that $\inf \kappa(x, y) = 0$ but that the same infimum taken on balls of increasing radii around some origin is non-zero. Is there a systematic correspondence between the way curvature decreases to 0 at infinity and the isoperimetric profile? (Compare the section of [Oll09] devoted to the relationship between non-negative Ricci curvature and exponential concentration.) An interesting example is the $M/M/k$ queue.

Problem I (Local assumptions for concentration).
In the Gaussian concentration theorem, the condition $\sigma_\infty < \infty$ can be replaced with a local Gaussian-type assumption, namely that for each measure m_x there exists a number s_x such that $\mathbb{E}_{m_x} \mathrm{e}^{\lambda f} \leqslant \mathrm{e}^{\lambda^2 s_x^2/2} \mathrm{e}^{\lambda \mathbb{E}_{m_x} f}$ for any 1-Lipschitz function f. Then a similar theorem holds, with $\sigma(x)^2/n_x$ replaced with s_x^2. (When s_x^2 is constant this is Proposition 2.10 in [DGW04].) However, this is not at all well-suited to discrete settings, because when transition probabilities are small, the best s_x^2 for which such an inequality is satisfied is usually much larger than the actual variance $\sigma(x)^2$: for example, if two points x and y are at distance 1 and $m_x(y) = \varepsilon$, s_x must satisfy $s_x^2 \geqslant 1/2 \ln(1/\varepsilon) \gg \varepsilon$. Thus making this assumption will provide extremely poor estimates of the variance D^2 when some transition probabilities are small (e.g. for binomial distributions on the discrete cube). In particular, when taking a continuous-time limit as above, such estimates diverge. So, is there a way to relax the assumption σ_∞, yet keep an estimate based on the local variance σ^2, and can this be done so that the estimate stays bounded when taking a continuous-time limit?

Problem J (Functional inequalities).
The Laplace transform estimate $\mathbb{E}\mathrm{e}^{\lambda(f-\mathbb{E}f)} \leqslant \mathrm{e}^{D^2\lambda^2/2}$ often used to establish Gaussian concentration for a measure ν is equivalent, by a result of Bobkov and Götze [BG99], to the following inequality: $\mathcal{T}_1(\mu,\nu) \leqslant \sqrt{2D^2 \ \mathrm{Ent}(\mathrm{d}\mu/\mathrm{d}\nu)}$ for any probability measure $\mu \ll \nu$. Is there a way to formulate our results in terms of functional inequalities? As such, the inequality above will fail as concentration can be non-Gaussian far away from the mean (e.g. in the simple example of the binomial distributions on the cube), so in a coarse setting it might be necessary to plug additive terms in the formulation of the inequality to account for what happens at small measures or small scales. Another suggestion by Villani is to use a Talagrand inequality where the L^2 transportation distance is replaced with a quadratic-then-linear transportation cost and use the results in [GL07].

Problem K (Sturm–Lott–Villani definition).
What is the relationship (if any) between our notion and the one defined by Sturm and Lott–Villani [Stu06, LV]? The latter is generally more difficult to work out on concrete examples, and is not so well suited to discrete settings (though see [BS]), but under the stronger $\mathrm{CD}(K,N)$ version, some more theorems are proven, including the Brunn–Minkowski inequality and Bishop–Gromov comparison theorem, together with applications to the Finsler case [Oht, OS].

Problem L (Bishop–Gromov theorem).
Is it possible to generalize more traditional theorems of positive Ricci curvature, i.e. the Bishop–Gromov theorem, or something close to the isoperimetric form of the Gromov–Lévy theorem? It is not clear what a reference constant curvature space would be in this context. Observe for example that, in the discrete cube, the growth of balls in exponential-like for small values of the radius (namely N, $N(N-1)/2$, etc.). Such theorems may be limited to manifold-like spaces for which a reference comparison space exists. Yet in the case of the cube, the isoperimetric behavior of balls still has something to do with that of the sphere and "slows down" in a positive-curvature-like way. A maybe useful definition of the "boundary" of a part A is $\mathcal{T}_1(1_A, 1_A * m)$. Also compare Problem R below.

Problem M (Entropy decay).
The logarithmic Sobolev inequalities (under the form comparing $\operatorname{Ent} f^2$ to $\int \|\nabla f\|^2$, not under the modified form comparing $\operatorname{Ent} f$ to $\int \|\nabla f\|^2 / f$) usually implies an exponential decreasing of entropy by the Markov chain. Is there some form of this phenomenon in our setting? (Once more, it is necessary to keep in mind the case of binomial distributions on the cube, for which the modified form of the Sobolev logarithmic inequality was introduced.)

Problem N (Discrete Ricci flow).
Define a "discrete Ricci flow" by letting the distance on X evolve according to Ricci curvature

$$\frac{\mathrm{d}}{\mathrm{d}t} d(x,y) = -\kappa(x,y)\, d(x,y)$$

where $\kappa(x,y)$ is computed using the current value of the distance (and by either keeping the same transition kernel m_x or having it evolve according to some rule). What can be said of the resulting evolution? (Note that if the same transition kernel is kept, then this will only compare to the usual Ricci flow up to a change of time, since, e.g. on a Riemannian sphere, this will amount to using smaller and smaller "diffusion constants" whereas the diffusion constant C in the Ricci flow $\frac{\mathrm{d}g}{\mathrm{d}t} = -C \operatorname{Ric}$ is taken constant; in particular, the diameter of a sphere will tend exponentially towards 0 instead of linearly.)

Problem O (Up to δ).
The constraint $\mathcal{T}_1(m_x, m_y) \leqslant (1-\kappa)\, d(x,y)$ may be quite strong when x and y are too close, even if the measures m_x, m_y have a larger support. In order to eliminate completely the influence of small scales, and in the spirit of δ-hyperbolic spaces, we can define a "positive curvature up to δ" condition. Namely, $\kappa(x,y)$ is the best $\leqslant 1$ constant in the inequality

$$\mathcal{T}_1(m_x, m_y) \leqslant (1-\kappa(x,y))\, d(x,y) + \delta$$

so that positive curvature up to some δ becomes an *open* property in Gromov–Hausdorff topology. Which theorems extend to this setting? Is it possible, in such a situation, to choose a discrete subset $X' \subset X$ and to redefine the random walk on X' in a reasonable way such that is has positive Ricci curvature?

Problem P (Discrete sectional curvature).
A notion equivalent to non-negative sectional curvature for Riemannian manifolds can be obtained by requiring that there be a coupling between m_x and m_y, such that the coupling moves *all* points of m_x by *at most* $d(x,y)$. (This amounts to replacing $\mathcal{T}_1$ with the L^∞ transportation distance in the definition.) Does this have any interesting properties? Is it possible to get an actual value for sectional curvature? (In this definition, the contribution from x and y themselves will generally prevent getting non-zero values.) Is this related to positive sectional curvature in the sense of Alexandrov? (Though the latter cannot be applied to discrete spaces.)

Problem Q (Discrete scalar curvature).
In Riemannian geometry, scalar curvature at x is the average of $\operatorname{Ric}(v)$ over all unit

vectors v around x. It controls, in particular, the growth of the volume of balls. Here one can transpose this definition and set $S(x) := \int \kappa(x,y)\,\mathrm{d}m_x(y)$ (where maybe a weight depending on $d(x,y)$ should be added). Has it any interesting properties?

Problem R (L^2 Bonnet–Myers and the dimension parameter).
For an L^2 version of the Bonnet–Myers theorem to hold, it is necessary to make stronger assumptions than positive curvature, namely that for any points x, x' and for any small enough pair of times t, t' one has

$$\mathcal{T}_1(m_x^{*t}, m_{x'}^{*t'}) \leqslant \mathrm{e}^{-\kappa \inf(t,t')} d(x,x') + \frac{C(\sqrt{t}-\sqrt{t'})^2}{2d(x,x')}$$

whereas before one used only the case $t = t'$. (The second term is obtained by considering Gaussian measures of variance t and t' centered at x and x' in $\mathbb{R}^N$.) Then (see the section on strong Bonnet–Myers theorem in [Oll09]) one gets a diameter estimate $\operatorname{diam} X \leqslant \pi\sqrt{\frac{C}{2\kappa}}$ so that C plays the role of $N-1$. Is the constant C somehow related to a "dimension", in particular to the "dimension" n in the Bakry–Émery $CD(K,n)$ condition?

Problem S (Alexandrov spaces).
What happens for spaces with positive sectional curvature in the sense of Alexandrov? Do they have positive Ricci curvature for a reasonable choice of m_x? (For the Sturm–Lott–Villani definition this has been proven in 2009 by Petrunin [Pet].) Would it be enough to approximate these spaces by manifolds or use a parallel transport in Alexandrov spaces? (See also Problem P above.)

Problem T (Expanders).
Is there a family of expanders (i.e. a family of graphs of bounded degree, spectral gap bounded away from 0 and diameter tending to ∞) with non-negative Ricci curvature? (Suggested by A. Naor and E. Milman.)

Problem U (Permutation groups).
For the permutation groups, with respect to the random walk generated by transpositions, Ricci curvature is positive but does not allow to recover concentration of measure with the correct order of magnitude. Is this related to results by N. Beresticky about the δ-hyperbolic-like properties of the permutation groups, which thus appear to have a mixture of positive and negative curvature properties?

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