RANDOM GROUPS UPDATE
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Abstract. This document is an attempt to keep track of developments in the study of random groups since the publication of *A January 2005 invitation to random groups*.

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Cayley graphs with expanders. In [AD], Goulnara Arzhantseva and Thomas Delzant give a full proof of Gromov's construction of a random group whose Cayley graph contains a family of expanders, and thus which does not coarsely embed into a Hilbert space. A nice summary of the context of this question is included. (The reader will need non-negligible prerequisites, notably some familiarity with Gromov's *mesoscopic curvature* and *very small cancellation theory* [Gro01, DG08].)

Actions on trees and boundary at infinity. In [DGP], François Dahmani, Vincent Guirardel and Piotr Przytycki prove that a random group G in the density model, at any positive density $0 < d < \frac{1}{2}$, enjoys the following properties:

- It has property FA (every action on a tree has a fixed point), and in particular it does not split.
- Its boundary at infinity is a Menger curve.
- $\mathrm{Out}(G)$ is finite.
- For any fixed torsion-free hyperbolic group Γ, $\mathrm{Hom}(G, \Gamma)$ is finite up to conjugacy.

The latter property, which relies on work by Z. Sela, can be nicely restated as follows: In a torsion-free hyperbolic group Γ, a system of (long) random equations has only a finite number of solutions up to conjugacy.

Critical density $d = \frac{1}{2}$. In a personal communication, Gady Kozma proves the following: For any $\varepsilon > 0$, consider a random group with m generators and $\varepsilon\,(2m-1)^{\ell/2}$ reduced relators of length ℓ; then with probability tending to 1 as $\ell \to \infty$, such a group is either $\{e\}$ or $\mathbb{Z}/2\mathbb{Z}$.

This covers the particular case $d = \frac{1}{2}$ of the density model, as well as the suggestion from [Oll05] (IV.a) to take $d \to \frac{1}{2}$ and $\ell = C/(\frac{1}{2} - d)$. However, the argument does not extend to the case of $\ell^{-\alpha}\,(2m-1)^{\ell/2}$ relators, with $\alpha > 0$.

Random complexes. In [BHK], Eric Babson, Christopher Hoffman and Matthew Kahle consider the fundamental group of random 2-complexes. A random simplicial 2-complex is obtained by considering a complete graph with n vertices, and adding each possible 2-cell at random, independently, with probability p. One of their results is that if $p \ll n^{-1/2}$ then the fundamental group of a random 2-complex is very probably non-trivial and hyperbolic, whereas if $p \gg n^{-1/2}$ it is very probably trivial.

This results bears a strong resemblance to Gromov's density $\frac{1}{2}$ theorem, but as far as I know the analogy is not fully understood. Some of the tools are very similar (e.g. Gromov's local-global criterion for hyperbolic spaces).

In the same setting, the first homology group of the 2-complex with coefficients in a finite field vanishes much earlier than the fundamental group, with a critical value

$p = 2\log n/n$ [LM06]. This is consistent with the fact that random groups in positive density have a trivial abelianization.

These papers contain additional references for topological properties of random complexes.

Algebraic properties at small density. In [KP09a] Ilya Kapovich and Paul Schupp consider random quotients (in the few-relator model) of the modular group $\mathrm{PSL}_2(\mathbb{Z})$, and prove that most of the time these quotients are not mutually isomorphic, with additional restrictions on possible homomorphisms between them; the same rigidity property for random groups (i.e. random quotients of a free group instead of $\mathrm{PSL}_2(\mathbb{Z})$) is still a conjecture. In [KP09b] they extend several algebraic results obtained by Arzhantseva and Ol'shanskiĭ (rank, freeness of subgroups. . .) from the few-relator model of random groups to small densities in the density model, and observe that the small density is not uniform in the number of generators of the group.

Groups not acting on manifolds. In [FS09], David Fisher and Lior Silberman prove that if a finitely generated group has no finite quotients and has the fixed point property for actions on non-positively curved Hilbert manifolds (i.e. complete geodesic CAT(0) spaces all of whose tangent cones are Hilbert spaces), then this group does not act non-trivially by measure-preserving homeomorphisms on a compact manifold. As discussed in this paper, random groups are an important source of examples of groups satisfying the assumptions; more precisely, direct limits of an infinite number of successive random quotients starting with a suitable (random or non-random) group.

Fixed point properties for actions on CAT(0) spaces. In [IKN09], Hiroyasu Izeki, Takefumi Kondo and Shin Nayatani consider actions of groups on CAT(0) spaces and study conditions under which every isometric action of a given group on such a space has a fixed point (in the spirit of property (T)). In particular, they prove that random groups in the sense of Żuk's triangular model, at density $d > 1/3$, have the fixed point property for actions on a particular class of CAT(0) spaces (those with bounded singularities in some sense).

Since the fixed point property is stable under quotients, it should pass from the triangular model to the ordinary density model (see discussion in § I.3.g of [Oll05]).

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