On a small cancellation theorem of Gromov

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Abstract

We give a combinatorial proof of a theorem of Gromov, which extends the scope of small cancellation theory to group presentations arising from labelled graphs.

In this paper we present a combinatorial proof of a small cancellation theorem stated by M. Gromov in [Gro03], which strongly generalizes the usual tool of small cancellation. Our aim is to complete the six-line-long proof given in [Gro03] (which invokes geometric arguments).

Small cancellation theory is an easy-to-apply tool of combinatorial group theory (see [Sch73] for an old but nicely written introduction, or [GH90] and [LS77]). In one of its forms, it basically asserts that if we face a group presentation in which no two relators share a common subword of length greater than 1/6 of their length, then the group so defined is hyperbolic (in the sense of [Gro87], see also [GH90] or [Sho91] for basic properties), and infinite except for some trivial cases.

The theorem extends these conclusions to much more general situations. Suppose that we are given a finite graph whose edges are labelled by generators of the free group F_m and their inverses (in a reduced way, see definition below). If no word of length greater than 1/6 times the length of the smallest loop of the graph appears twice on the graph, then the presentation obtained by taking as relations all the words read on all loops of the graph defines a hyperbolic group which (if the rank of the graph is at least $m+1$, to avoid trivial cases) is infinite. Moreover, the given graph naturally embeds isometrically into the Cayley graph of the group.

The new theorem reduces to the classical one when the graph is a disjoint union of circles. Noticeably, this criterion is as easy to use as the standard one.

For example, ordinary small cancellation theory cannot deal with such simple group presentations as $\langle\, S \mid w_1 = w_2 = w_3 \,\rangle$ because the two relators involved here, $w_1 w_2^{-1}$ and $w_1 w_3^{-1}$, share a long common subword. The new theorem can handle such situations: for "arbitrary enough" words w_1, w_2, w_3, such presentations will define infinite, hyperbolic groups, although from the classical point of view these presentations satisfy (e.g. if the w_i's have the same length) a priori only the $C'(1/2)$ condition from which nothing could be deduced.

The groups obtained by this process can in some cases be noticeably different from ordinary small cancellation groups. For example, the graphs used by Gromov in [Gro03] provide groups having Kazhdan's property (T) (see [Sil03]), whereas ordinary small cancellation groups cannot have property (T) (see [Wis04]).

Most importantly, this technique allows to (quasi-)embed prescribed graphs into the Cayley graphs of hyperbolic groups. It is the basic construction involved in the announcement of a counter-example to the Baum-Connes conjecture with coefficients (see [HLS02] which elaborates on [Gro03], or [Ghy03] for a survey). Indeed, this counter-example is obtained by constructing a finitely generated group (which is a limit of hyperbolic groups) whose Cayley graph quasi-isometrically contains an infinite family of expanders.

Moreover, this technique will be used in [OW04] to construct new examples of groups with property (T).

1 Statement and discussion

Let S be a finite set, in which an involution without fixed point, called *being inverse*, is given. The elements of S are called *letters*.

A *word* is a finite sequence of letters. The inverse of a word is the word made of the inverse letters put in reverse order. A word is called *reduced* if it does not contain a letter immediately followed by its inverse.

A *labelled graph* is an unoriented graph in which each unoriented edge is considered as a couple of two oriented edges, and each oriented edge bears a letter such that opposite edges bear inverse letters. We require maps of labelled graphs to preserve the labels.

A labelled graph is said to be *reduced* if there is no pair of oriented edges arising from the same vertex and bearing the same letter.

Note that a word can be seen as a (linear) labelled graph, which we will implicitly do from now on. The word is reduced if and only if the labelled graph is.

A *piece* of a labelled graph is a word which has two different immersions in the labelled graph. (An immersion is a locally injective map of labelled graphs. Two immersions are considered different if they are different as maps.) This is analogue to the traditional piece of small cancellation theory.

A *standard family of cycles* for a connected graph is a set of paths in the graph, generating the fundamental group, such that there exists a maximal subtree of the graph such that, when the subtree is contracted to a point (so that the graph becomes a bouquet of circles), the set of generating cycles is exactly the set of these circles. There always exists some. If the graph is not connected, a standard family of cycles is one which is standard on each component.

A *generating family of cycles* is a family of cycles generating the fundamental group of each connected component of the graph (maybe up to adding initial and final segments joining these cycles to some basepoint).

A graph is *non-filamenteous* if every edge belongs to some immersed cycle.

We are now in a position to state the theorem.

Theorem 1 (M. Gromov, [Gro03]). *Let Γ be a finite reduced non-filamenteous labelled graph. Let R be the set of words read on all cycles of Γ (or on a generating family of cycles). Let g be the girth of Γ and Λ be the length of the longest piece of Γ.*

If $\Lambda < g/6$ then the presentation $\langle\, S \mid R \,\rangle$ defines a group G enjoying the following properties.

1. *It is hyperbolic, torsion-free.*

2. *Any presentation of G by the words read on a standard family of cycles of Γ is aspherical (in the sense of Definition 9), hence the cohomological dimension of G is at most 2.*

3. *The Euler characteristic of G is $\chi(G) = 1 - |S|/2 + b_1(\Gamma)$. In particular, if the rank of the fundamental group of Γ is greater than the number of generators, G is infinite and not quasi-isometric to $\mathbb{Z}$.*

4. *The shortest relation in G is of length g.*

5. *For any reduced word w representing the identity in G, some cyclic permutation of w contains a subword of a word read on a circle immersed in Γ, of length at least $(1 - 3\Lambda/g)$ (which is more than $1/2$) times the length of this cycle.*

6. *The natural maps from each connected component of the labelled graph Γ into the Cayley graph of G are isometric embeddings.*

If Γ is a disjoint union of circles, this theorem almost reduces to ordinary $1/6$ small cancellation theory. The "almost" accounts for the fact that the length of a shared piece between two relators is supposed to be less than $1/6$

the length of the smallest of the two relators in ordinary small cancellation theory, and less than 1/6 the length of the smallest of all relators in our case; this is handled through the following remark (which we do not prove in order not to have still heavier notation).

Remark 2. It is clear from the proof that the assumption in the theorem can be replaced by the following slightly weaker one: for each piece, its length is less than 1/6 times the length of any cycle of the graph on which the piece appears.

With this latter assumption, the theorem reduces to ordinary small cancellation when the graph is a disjoint union of circles.

Remark 3. Non-filamenteousness is needed only to ensure isometric embedding of the graph (filaments may not embed isometrically if $\Lambda \geqslant g/8$).

The group obtained is not always non-elementary: for example, if there are three generators a, b, c and the graph consists in two points joined by three edges bearing a, b and c respectively, one obtains the presentation $\langle\, a, b, c \mid a = b = c \,\rangle$ which defines $\mathbb{Z}$. However, since the cohomological dimension is at most 2, it is easy to check (computing the Euler characteristic) that if the rank of the fundamental group of Γ is greater than the number of generators, then G is non-elementary.

This theorem is not stated explicitly in [Gro03] in the form we give but using a much more abstract and more powerful formalism of "rotation families of groups" ([Gro03], section 2). In the vocabulary thereof, the case presented here is when this rotation family contains only one subgroup of the free group (and its conjugates), namely the one generated by the words read on cycles of the graph with some base point; the corresponding "invariant line" U is the universal cover of the labelled graph Γ (viewed embedded in the Cayley graph of the free group). Reducedness of the labelling ensures convexity.

Elements for a proof of the theorem for very small values of Λ/g (instead of $\Lambda/g < 1/6$) using geometric rather than combinatorial tools, can be found in [Gro01] (see also [Gro03], p. 88).

In [Gro03], this theorem is applied to a random labelling (or rather a variant, Theorem 18 below, in which reducedness is replaced with quasi-geodesicity). It is not difficult, using for example the techniques described in [Oll04], to check that a random labelling satisfies the small cancellation and quasi-geodesicity assumptions.

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2 Idea of proof

The line of the argument is as follows: Choose a presentation of G by the words read on a standard generating family of cycles of Γ. We will study the isoperimetry of van Kampen diagrams with respect to this set of relations: we will show that the number of faces in such diagrams is linearly bounded by its boundary length.

Define a labelled complex Γ_2 by attaching to Γ a disk for each cycle in the family. Now each face of a van Kampen diagram for this presentation can be lifted (in a unique way) to Γ_2. For any edge between two faces of the diagram, either these two faces are already adjacent along "the same" edge in Γ_2 or they are not.

Decompose the diagram into maximal parts all edges of which originate from Γ_2 in this sense. Now gluings between these parts do not originate from Γ_2 and thus constitute pieces. So these parts are in classical 1/6 small cancellation with respect to each other, and so the boundary length of the diagram is controlled in terms of the boundary lengths of these parts. We get the other usual consequences of small cancellation theory as well (asphericity, radius of injectivity...). Technicalities arise from the necessity to perform some so-called "diamond moves" and from the maybe non-simple connectedness of these parts.

To reach the conclusion it is then enough to work inside each part. Since each part lifts to Γ_2 its boundary word is the word read on some null-homotopic cycle in Γ_2. So this cycle is the product of elements our generating family of cycles, and for isoperimetry we have to control the number of terms in this product (the number of faces in the part) in function of the length of the cycle (the boundary length of the part). This is achieved by decomposing the considered cycle into a product of cycles shorter than three times the diameter of the graph. As there are only finitely many such short cycles we are done.

3 Proof (expanded version)

We now give some more definitions which are useful for the proof.

Definition 4. *A* labelled complex *is a finite unoriented combinatorial*

2-complex the interior of every face of which is homeomorphic to an open disk D_{n+1} with $n \geqslant 0$ holes (n depends on the face), such that its 1*-skeleton is equipped with a labelled graph structure.*

A labelled complex is said to be reduced *if its* 1*-skeleton is.*

Each face of such a complex defines a set of *contour words*: If the interior of the face is homeomorphic to an open disk D_{n+1} with n holes, the contour words are the $n+1$ cyclic words read by moving around the $n+1$ boundary components of D_{n+1}. The words in this set are considered as oriented cyclic words, and counted with multiplicities.

We require a map of labelled complexes to preserve labels (but it may change orientation of faces, sending a face to a face with inverse contour labels — this amounts to considering maps between the corresponding oriented complexes).

Definition 5. *A* tile *is a planar labelled complex with only one face (not necessarily simply connected) and each edge of which belongs to the combinatorial boundary of the face with multiplicity one. We do not fix the embedding in the plane.*

It follows from the definition that the contour of a tile coincides with its boundary.

By our definition of maps between labelled complexes, a tile is considered equal to the tile bearing the inverse boundary words.

Convention: A tile may bear a word which is not simple (i.e. is a power of a smaller word). In this case the tile would have a non-trivial automorphism. To prevent this, say that on each boundary component of a tile we mark a starting point and that a map between tiles has to preserve marked points. This is useful for the study of asphericity and torsion (see Definition 9).

To any planar labelled complex with only one face we can associate a tile in the following way: First, remove the edges that do not belong to the adherence of the interior of the complex (the "filaments"). Then, the obtained one-face complex immersed in the plane is the image of some one-face complex embedded in the plane by a cellular map (this complex is constructed by ungluing along the internal edges). This is an embedding in the plane of some tile, which we call the *tile associated to* the one-face labelled complex.

Definition 6. *A* tile of a labelled complex *is the tile associated to any of its faces.*

The *length* of a tile is the length of its boundary.

Definition 7. *A* piece *with respect to a set of tiles is a word which has immersions in the boundary of two different tiles, or two distinct immersions in the boundary of one tile.*

Definition 8. *A* puzzle *with respect to a set of tiles is a planar labelled complex all tiles of which belong to this set of tiles (the same tile may appear several times in a puzzle). The* set of boundary words *of a puzzle is the set of words read on its boundary components (with multiplicities and orientations).*

A spherical puzzle *is the same drawn on a sphere instead of the plane, that is, a labelled complex which is a combinatorial 2-sphere, all tiles of which belong to this set of tiles.*

A puzzle is said to be minimal *if it has the minimal number of tiles among all puzzles having the same set of boundary words.*

A puzzle is said to be van Kampen-reduced *if there is no pair of adjacent faces such that the words read on the external contour of these two faces are inverse and the position (with respect to the marked point) of the letter read at a common edge of these faces is the same in the two copies of the contour word of these faces.*

So a puzzle is roughly speaking a van Kampen diagram in which we allow non-simply connected faces. The last definition given corresponds to reduced van Kampen diagrams (see [LS77]). (Incidentally, a reduced puzzle is van Kampen-reduced, though the converse is not necessarily true.)

Definition 9. *A presentation of a group is said to be* aspherical *if the set of tiles whose boundary words are the relators of the presentation admits no van Kampen-reduced spherical puzzle.*

There are several notions of aspherical presentations in the literature (see e.g. [CCH81] for five of them). Our definition of asphericity coincides with the one in [Ger87], p. 31 (in which asphericity is termed "every spherical diagram is diagrammatically reducible"). It is thus stronger than the one(s) in [LS77], the main difference being that we mark a starting point on the boundary of each tile (see the discussion in [Ger87]). In particular, with our (and [Ger87]'s, contra [LS77]) convention, a presentation such as $\langle\, S \mid w^n = 1 \,\rangle$ (with $n \geqslant 2$) is not aspherical: no relator can be a proper power. With this convention, asphericity of a presentation implies asphericity of the Cayley 2-complex ([Ger87], p. 32), hence (by Hurewicz' Theorem) cohomological dimension at most 2 and hence ([Bro82], p. 187) torsion-freeness.

Proof of the theorem.

Let Γ be a reduced labelled graph. The group under consideration is defined by the presentation $\langle S \mid R \rangle$ where R is the set of all words read along cycles of Γ. However, taking all words is not necessary: the group presented by $\langle S \mid R \rangle$ will be the same if we take not all cycles but only a generating set of cycles.

The fundamental group of the graph Γ is a free group. Let $\mathcal{C}$ be a finite generating set of $\pi_1(\Gamma)$ (maybe not standard). Let R be the set of words read on the cycles in $\mathcal{C}$.

Add 2-faces to Γ in the following way: for each cycle in $\mathcal{C}$, glue a disk bordering this cycle. Denote by Γ_2 this 2-complex; it depends on the choice of $\mathcal{C}$, or equivalently on R.

As the cycles in $\mathcal{C}$ generate all cycles, Γ_2 is simply connected. Note that if $\mathcal{C}$ happens to be taken standard, as will sometimes be the case below, then Γ_2 has no homotopy in degree 2.

By our definitions above (Definition 6), a tile of Γ_2 is a topological disk whose boundary is labelled by some word of R.

We are going to show that there exists a constant $C > 0$ such that any simply connected van Kampen-reduced puzzle D with respect to the tiles of Γ_2 satisfies a linear isoperimetric inequality $|\partial D| \geqslant C\,|D|$ where $|\partial D|$ is the boundary length of D and $|D|$ is the number of faces of D. This implies hyperbolicity (see for example [Sho91]).

We can safely assume that all edges of D lie on the contour of some face (roughly speaking, there are no "filaments"). Indeed, filaments only improve isoperimetry. Generally speaking, in what follows we will never mention the possible occurrence of filaments, their treatment being immediate.

Remark 10. The 1/6 assumption on pieces implies that no two distinct cycles of Γ bear the same word.

Let e be an internal[1] edge of D, adjacent[2] to faces f_1 and f_2. As D is a puzzle over the tiles of Γ_2, there are faces f_1' and f_2' of Γ_2 bearing the same contour words as f_1 and f_2 respectively (maybe up to inversion). These faces are unique by Remark 10.

The edge e belongs to the contour of both f_1 and f_2 and thus can be lifted in Γ_2 either in f_1' or in f_2'. Say e is an *edge originating from* Γ_2 if these two lifts coincide, so that in Γ_2, the two faces at play are adjacent along the same edge as they are in D.

[1] i.e. not on the boundary

[2] We say that two faces f_1, f_2 of a 2-complex are *adjacent* along edge e (or simply *adjacent* if the mention of e is unnecessary) if either $f_1 \neq f_2$ and e belongs to the contour of both f_1 and f_2, or $f_1 = f_2$ and e is included twice in the contour of f_1.

Any labelled complex with respect to the tiles of Γ_2, all internal edges of which originate from Γ_2, can thus be lifted to Γ_2 by lifting each of its edges. This lifting is unique by Remark 10.

Note that D is van Kampen-reduced if and only if there is no edge e originating from Γ_2 and adjacent to faces f_1, f_2 such that $f'_1 = f'_2$.

We work by first proving the isoperimetric inequality for puzzles having all edges originating from Γ_2. Second, we will decompose the puzzle D into "parts" having all their edges originating from Γ_2 and show that these parts are in $1/6$ small cancellation with each other. Then we will use ordinary small cancellation theory to conclude.

We begin by proving what we want for some particular choice of R.

Lemma 11. *Let $\Delta = \mathrm{diam}(\Gamma)$. Suppose that $\mathcal{C}$ was chosen to be the set of closed paths embedded (or immersed) in Γ of length at most 3Δ. Then, for any closed path in Γ labelling a reduced word w, there exists a simply connected puzzle with boundary word w, with tiles having their boundary words in R, all edges of which originate from Γ_2, and with at most $3\,|w|\,/g$ tiles.*

Proof of Lemma 11.
If $|w| \leqslant 2\Delta$ then by definition of R there exists a one-tile puzzle spanning w, and as $|w| \geqslant g$ the conclusion holds. Show by induction on n that if $|w| \leqslant n\Delta$ there exists a simply connected puzzle D spanning w with at most n tiles. This is true for $n = 2$. Suppose this is true up to $n\Delta$ and suppose that $2\Delta \leqslant |w| \leqslant (n+1)\Delta$.

Let $w = w'w''$ where $|w'| = 2\Delta$. As the diameter of Γ is Δ, there exists a path in Γ labelling a word x joining the endpoints of w', with $|x| \leqslant \Delta$. So $w'x^{-1}$ is read on a cycle of Γ of length at most 3Δ, hence (its reduction) belongs to R. Now xw'' is a word read on a cycle of Γ, of length at most $|w| - \Delta \leqslant n\Delta$. So there is a puzzle with at most n tiles spanning xw''. Gluing this puzzle with the tile spanning $w'x^{-1}$ along the x-sides provides the desired puzzle. (Note that this gluing occurs in Γ_2, so that edges of the resulting puzzle originate from Γ_2.)

So for any w we can find a puzzle spanning it with at most $1 + |w|\,/\Delta$ tiles. As $\Delta \geqslant g/2$ and as $|w| \geqslant g$, we have $1 + |w|\,/\Delta \leqslant 1 + 2\,|w|\,/g \leqslant 3\,|w|\,/g$. $\square$

Corollary 12. *For any choice of $\mathcal{C}$, there exists a constant $\alpha > 0$ such that any minimal simply connected puzzle D with respect to the tiles of Γ_2 all internal edges of which originate from Γ_2 satisfies the isoperimetric inequality $|\partial D| \geqslant \alpha\,|D|$.*

Proof of Corollary 12.
Indeed, the existence of an isoperimetric constant for minimal diagrams does not depend on the finite presentation, hence the result when $\mathcal{C}$ is finite. This also holds for infinite $\mathcal{C}$ since any infinite family of cycles in the finite graph Γ contains a finite generating subfamily. $\square$

These last affirmations only express in terms of diagrams the fact that the fundamental group of Γ, which is free hence hyperbolic, is generated by the cycles of Γ of length at most 3Δ (w.r.t. some basepoint).

The next lemma is just ordinary small cancellation theory (see for example the appendix of [GH90], or [LS77]), stated in the form we need. Note that usually, the definition of small cancellation involves pieces of relative size less than λ with $\lambda \leqslant 1/6$. Here we use pieces of relative size at most λ with $\lambda < 1/6$. This is less well-suited for treatment of infinite presentations (which we do not consider) but allows lighter notation for the isoperimetric constant $1-6\lambda > 0$ and the Greendlinger constant $1-3\lambda > 1/2$.

Lemma 13. *Let R be a set of simply connected reduced tiles. Suppose that any piece with respect to two tiles $t, t' \in R$ is a word of length at most λ times the smallest boundary length of t and t', for some constant $\lambda < 1/6$.*

Then any simply connected van Kampen-reduced puzzle D with respect to the tiles of R satisfies the following properties.

1. *If D has at least two faces, the reduction w of the boundary word of D contains two disjoint subwords w_1, w_2, with w_1 (resp. w_2) subword of the boundary word of some tile t_1 (resp. t_2) of D, with length at least $(1-3\lambda) > \frac{1}{2}$ times the boundary length of t_1 (resp. t_2).*

2. *The word w is not a proper subword of the boundary word of some tile.*

3. *The boundary length $|\partial D|$ is at least $1-6\lambda$ times the sum of the lengths of the faces of D, and at least the boundary length of the largest tile it contains.*

Moreover, there is no spherical van Kampen-reduced puzzle with respect to these tiles.

Corollary 14. *Let R be a set of (not necessarily simply connected) reduced tiles. Suppose that any piece with respect to two tiles $t, t' \in R$ is a word of length at most λ times the smallest length of the boundary component of t and t' it immerses in, for some constant $\lambda < 1/6$.*

Then, any simply connected puzzle with respect to this set of tiles contains only simply connected tiles.

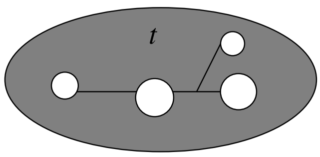
Proof of the corollary.

Let D be a simply connected puzzle with respect to R. Let t be a non-simply connected tile in D. We can suppose that t is deepest, that is, that the bounded components of the complement of t contain no other non-simply connected tile.

The interior of t is embedded in the plane and is homeomorphic to a disk with some finite number n of holes. Since D is simply connected any such hole is filled with a subpuzzle. So let $D'_1, \ldots, D'_n$ be the subpuzzles filling the bounded connected components of the complement of the interior of t. Each D'_i is simply connected, since the bounded connected components of the complement of a connected set in the plane are simply connected. Let us work with D'_1. In case D'_1 is not van Kampen-reduced we replace it by its van Kampen-reduction (which does not change its boundary word, so it can still be glued to one of the holes of t).

The boundary of D'_1 may not be embedded in the plane. However, it is immersed, since the word read on it is the word read on one of the interior boundaries of t, and this word is reduced.

The component D'_1 is a connected simply connected puzzle. Its image in the plane is the union of closed sets $D''_1, \ldots, D''_q$ such that each D''_i is either a topological closed disk or a topological closed segment ("filament"), and the D''_i's intersect at a finite number of points. By construction, each D''_i which is a disk is a puzzle.



Suppose that D''_i is a segment. Then each of its endpoints belongs to some D''_j with $j \neq i$. Indeed, otherwise the boundary of D'_1 would not be immersed.

Construct a graph T embedded in the plane in the following way. For each D''_i which is a disk, define a family of segments T_i as follows: Choose a point p_0 in the interior of D''_i. There are a finite number of points $p_1, \ldots, p_r$ on the boundary of D''_i such that p_j belongs to some D''_k for $k \neq i$. Now define T_i to be made of the union of segments $p_0p_j \subset D''_i$ for $1 \leqslant j \leqslant r$. Now define T to be the union of all D''_i for those $1 \leqslant i \leqslant q$ for which D''_i is a segment, plus the union of all T_i's for those $1 \leqslant i \leqslant q$ for which D''_i is a disk.

By construction, T is connected since D'_1 is.

For each i such that D''_i is a disk, D''_i retracts onto T_i preserving the points $p_1, \ldots, p_r$. So D'_1 retracts onto T, and in particular T is simply connected since D'_1 is. So T is a tree. It is non-empty since D'_1 is (but maybe reduced to a point if D'_1 is a topological disk).

Now consider some leaf of T. Since any endpoint of any D''_i which is a segment belongs to some D''_j with $j \neq i$ (since $\partial D'_1$ is immersed as we saw above), a leaf of T cannot belong to a D''_i which is a segment. So a leaf of T belongs to some T_i constructed from some D''_i which is a disk. By definition of T_i, this means that D''_i intersects with at most one other D''_j with $j \neq i$.

Now D''_i is a puzzle which is a topological disk. As we supposed that t was taken a deepest non-simply connected tile, D''_i contains only simply connected tiles. So we can apply Lemma 13: there exist two tiles t', t'' in D''_i and two subwords w', w'' of the boundary word of D''_i such that w' (resp. w'') is a subword of the boundary word of t' (resp. t'') of length at least one half the boundary length of t' (resp. t''). As D''_i has at most one point of intersection with the other D''_j for $j \neq i$, at least one of w' and w'' is a subword of the boundary of D'_1. But a boundary word of D'_1 is a boundary word of the tile t, and so t shares with t' or t'' a word of length at least one half the boundary length of t' or t'', which contradicts the small cancellation assumption. $\square$

Back to our simply connected van Kampen-reduced minimal puzzle D with tiles in Γ_2. A puzzle is built by taking the disjoint union of all its tiles and gluing them along the internal edges.

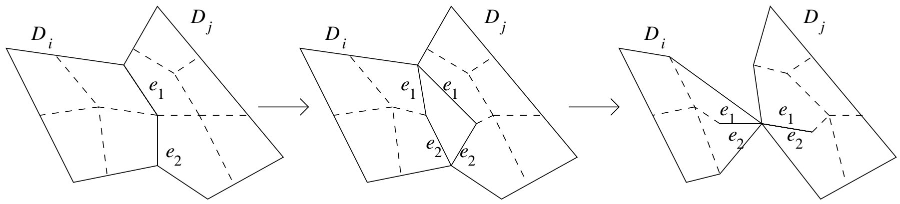
First, define a disjoint union of puzzles D' by taking the disjoint union of all tiles of D and gluing them along the internal edges of D originating from Γ_2. All internal edges of D' originate from Γ_2.

As D is van Kampen-reduced, D' is as well.

Let D_i, $i = 1, \ldots, n$ be the connected components of D'. They form a partition of D. The puzzle D is obtained by gluing these components along the internal edges of D not originating from Γ_2.

It may be the case that the boundary word of some D_i is not reduced. This means that there is a vertex on the boundary of D_i which is the origin of two (oriented) edges bearing the same vertex. We will modify D in order to avoid this. Suppose some D_i has non-reduced boundary word and consider two edges e_1, e_2 of D responsible for this: e_1 and e_2 are two consecutive edges with inverse labels. These edges are either boundary edges of D or internal edges. In the latter case this means that D_i is to be glued to some D_j. We treat only this latter case as the other one is even simpler.

Make the following transformation of D: do not glue any more edge e_1 of D_i with edge e_1 of D_j, neither edge e_2 of D_i with edge e_2 of D_j, but rather glue edges e_1 and e_2 of D_i, as well as edges e_1 and e_2 of D_j, as in the following picture. This is possible since by definition e_1 and e_2 bear inverse labels.



This kind of operation has been studied and termed *diamond move* in [CH82]. The case when the central point has valency greater than 2 (i.e. when more than two D_i's meet at this point) is treated similarly.

Since Γ_2 is reduced, the lifts to Γ_2 of the edges e_1 and e_2 of D_i are the same edge of Γ_2. This shows that the transformation above preserves the fact that all edges of D_i and of D_j originate from Γ_2.

The resulting puzzle (denoted D again) has the same number of faces as before, and no more boundary edges. Thus, proving isoperimetry for the modified puzzle will imply isoperimetry for the original one as well. So we can safely assume that the boundary words of the D_i's are reduced.

Now consider D as a puzzle with the D_i's as tiles. (More precisely, if we erase from D all internal edges originating from Γ_2 then we obtain a puzzle each tile of which is the tile associated to the one-face complex obtained from some D_i by erasing all internal edges originating from Γ_2.) This is a van Kampen-reduced puzzle, since if D_i and D_j are in reduction position this means that they lift to the same subcomplex of Γ_2 and share an edge originating from Γ_2, which contradicts their definition. Note that these tiles are not necessarily simply connected.

These tiles satisfy the condition of Corollary 14. Indeed, suppose that two tiles D_i, D_j (with maybe $i = j$ in which case two parts of the boundary of the same tile are glued) are to be glued along a common (reduced!) word w. By definition of the D_i's, the edges making up w do not originate from Γ_2.

As the edges of D_i originate from Γ_2, there is a lift $\varphi_i : D_i \to \Gamma_2$ (as noted above). Consider the two lifts $\varphi_i(w)$ and $\varphi_j(w)$. As the edges making up w do not originate from Γ_2, these two lifts are different. As w is reduced these lifts are immersions. So w is a piece. By assumption the length of w is at most $\Lambda < g/6$.

Now as D_i lifts to Γ_2, any boundary component of D_i goes to a closed path in Γ. This proves that the length of any boundary component of D_i is at least g.

So the tiles D_i satisfy the small cancellation condition with $\lambda = \Lambda/g < 1/6$. As they are tiles of a simply connected puzzle, by Corollary 14 they are simply connected.

Then by Lemma 13, the boundary of D is at least $1 - 6\lambda$ times the sum of the boundary lengths of the D_i's (considered as tiles). Since D is minimal, each D_i is as well, and as D_i is simply connected, by Corollary 12 it satisfies the isoperimetric inequality $|\partial D_i| \geqslant \alpha\,|D_i|$. So

$$|\partial D| \geqslant (1-6\lambda)\sum|\partial D_i| \geqslant \alpha(1-6\lambda)\sum|D_i| = \alpha(1-6\lambda)\,|D|$$

which shows the isoperimetric inequality for D, hence hyperbolicity.

For asphericity and the cohomological dimension (hence torsion-freeness), suppose that $\mathcal{C}$ is standard (so that Γ_2 is contractible) and that there exists a van Kampen-reduced spherical puzzle D, which we can assume to be inclusion-minimal in the sense that it contains no spherical subpuzzle. Define the D_i's as above. Either some D_i is spherical, in which case $D = D_i$ by inclusion-minimality of D, or all D_i's have non-empty boundary words. The former is ruled out by the following lemma:

Lemma 15. *Suppose that the set of paths read along faces of Γ_2 is standard. Let D be a non-empty spherical puzzle all edges of which originate from Γ_2. Then D is not van-Kampen reduced.*

Proof of the lemma.
Let T be a maximal tree of Γ witnessing for standardness of the family of cycles. Homotope T to a point. This turns Γ_2 into a bouquet of circles with a face in each circle. Similarly, homotope to a point any edge of D coming from a suppressed edge of Γ. This way we turn D into a spherical van Kampen diagram with respect to the presentation of the fundamental group of Γ_2 (i.e. the trivial group) by $\langle\, c_1, \ldots, c_n \mid c_1 = e, \ldots, c_n = e \,\rangle$. But there is no reduced spherical van Kampen diagram with respect to this presentation, as can immediately be checked. □

Since by definition each D_i lifts to Γ_2 and since D (hence each D_i) is van Kampen-reduced, the lemma implies that no D_i is spherical. Hence the D_i's have non-trivial boundary words. So D can be viewed as a spherical puzzle with the boundary words of the D_i's as tiles. But we saw above that the D_i's (viewed as tiles) satisfy the small cancellation condition. So by Lemma 13 there is no spherical van Kampen-reduced puzzle w.r.t. these tiles.

The computation of the Euler characteristic immediately follows, using that the cohomological dimension is at most 2.

The last assertions of the theorem follow easily from the assertions of Lemma 13. The smallest relation in the group presented by $\langle\, S \mid R \,\rangle$ is the boundary length of the smallest non-trivial puzzle, which by Lemma 13 is at least the smallest boundary length of the D_i's, which is at least the girth g. Similarly, any reduced word representing the trivial element in the group is read on the boundary of a van-Kampen reduced simply connected puzzle, thus contains as a subword at least one half of the boundary word of some D_i.

(Note: The version of this text published in Bull. Belg. Math. Soc. contains a mistake in this part of the argument, as it used that x' was geodesic. This was pointed to me by Mikhail Ostrovskii. Below is a corrected version.)

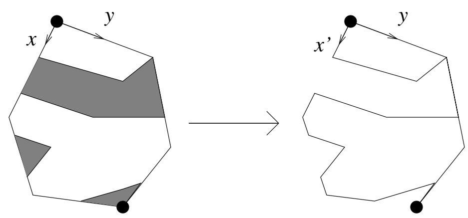
For the isometric embedding of Γ in the Cayley graph of the group, suppose that some geodesic path p in the graph (or in Γ_2) labelling a word x is equal to a shorter word y in the quotient. This means that there exists a van Kampen-reduced puzzle D with boundary word xy^{-1}, made up of tiles with cycles of Γ as boundary words.

Now x is the word read on a path p_1 in the boundary word of D, which lifts to the geodesic path p in Γ_2 labelling x as well. Let f be a face of D which intersects p_1 along at least one edge. We say that f *originates from* Γ_2 *together with* x if the lift from f to Γ_2 coincides with the lift $p_1 \to p$ on the intersection of f with p_1.

We are going to recursively remove all faces of D which originate from Γ_2 together with x, as follows. Let f be such a face of D, and assume it shares an edge e along with x, so that $x = x_1 e x_2$ and the boundary of D is ew. Define the path $x' = x_1 w^{-1} x_2$ in the diagram D, and remove face f from the diagram D. This defines a new diagram D' with boundary $x'y^{-1}$. Note that x' may not be reduced.

By construction, x' lifts to Γ_2 together with x, and their lifts to Γ_2 have the same endpoints. In particular, $|x'| \geqslant |x|$ since x is geodesic in Γ.

Repeat this process until there are no faces of D that originate from Γ_2 together with x'. At the end of this process, we still have that x' lifts to Γ_2 together with x, and $|x'| \geqslant |x|$. In the following picture in which black cells represent tiles originating from Γ_2 together with x.



At this point, if $x' = y$ (there are no faces left in the diagram), we are done, because $|y| = |x'| \geqslant |x|$ as needed.

If some faces are left, we have a (possibly non-reduced) puzzle with boundary $x'y^{-1}$. By construction, no faces of D' lift to Γ_2 together with x'. This means that the intersection of x' with any face of D' is a *piece* in Γ.

By assumption, y was reduced, but there may be cancellations within x' or between x' and y. Reduce D', first by removing any filaments in x', then by "folding in" any inverse consecutive edges of $x'y^{-1}$ as in the "diamond moves" above. Let w_1 and w_2 be the common initial and final segments between x' and y (if any); after reduction, we are left with a puzzle D'' with boundary word $x''(y')^{-1}$, where $x' = w_1 x'' w_2$, $y = w_1 y' w_2$, and $x''(y')^{-1}$ is cyclically reduced.

Now (if Γ contains no filaments) x'' is part of some cycle of Γ labelled by $x''z$. As Γ_2 is simply connected, there is a van Kampen-reduced puzzle D_2 with boundary word $x''z$ and which globally lifts to Γ_2 (all its edges originate from Γ_2).

Define a new puzzle D''' by gluing D_2 and D'' along the word x''. This is a puzzle bordering zy'. It is van Kampen-reduced since D_2 and D'' are van Kampen-reduced and since there is no cancellation between D_2 and D'' (otherwise there would be a tile of D'' originating from Γ_2 together with x'').

Now consider, as above, the partition $D''' = \cup D_i'''$ where the D_i''' are maximal parts lifting to Γ_2. Since no tile of D'' adjacent to x'' originates from Γ_2 together with x'', D_2 is exactly one of the D_i'''.

By Lemma 13, the boundary length $|z|+|y'|$ of D''' is at least the boundary length of any D_i'''. In particular, it is at least the boundary length of D_2, which is $|z| + |x''|$. This proves that $|z| + |y'| \geqslant |z| + |x''|$, and therefore, $|y'| \geqslant |x''|$ so that $|y| \geqslant |x'| \geqslant |x|$, as needed.

This proves the theorem. □

4 Further remarks

Remark 16. The proof above gives an explicit isoperimetric constant when the set of relators taken is the set of all words read on cycles of the graph of length at most three times the diameter: in this case, any minimal simply connected puzzle satisfies the isoperimetric inequality

$$|\partial D| \geqslant g(1 - 6\Lambda/g)\,|D|\,/3$$

This explicit isoperimetric constant growing linearly with g (i.e. "homogeneous") can be very useful if one wants to apply such theorems as the local-global hyperbolic principle, which requires the isoperimetric constant to grow linearly with the sizes of the relators.

Remark 17. The assumption that Γ is reduced can be relaxed a little bit, provided that some quasi-geodesicity assumption is granted, and that the definition of a piece is emended.

Redefine a *piece* to be a couple of words (w_1, w_2) such that both immerse in Γ and such that $w_1 = w_2$ in the free group. The *length* of a piece (w_1, w_2) is the maximal length of w_1 and w_2.

There are trivial pieces, for example if $w_1 = w_2$ and both have the same immersion. However, forbidding this is not enough: for example, if a word of the form $aa^{-1}w$ immerses in the graph, then $(aa^{-1}w, w)$ will be a piece.

A *trivial piece* is a piece (w_1, w_2) such that there exists a path p in Γ joining the beginning of the immersion of w_1 to the beginning of the immersion of w_2 such that p is labelled with a word equal to e in the free group.

The new theorem is as follows.

Theorem 18 (M. Gromov). *Let Γ be a finite non-filamenteous labelled graph. Let R be the set of words read on all cycles of Γ (or on a generating family of cycles). Let g be the girth of Γ and Λ be the length of the longest non-trivial piece of Γ.*

Suppose that $\lambda = \Lambda/g$ is less than $1/6$.

Suppose that there exist a constant $A > 0$ such that any word w immersed in Γ of length at least L satisfies $\|w\| \geqslant A(|w| - L)$ for some $L < (1-6\lambda)g/2$.

Then the presentation $\langle\, S \mid R\,\rangle$ defines a hyperbolic, infinite, torsion-free group G, and (if R arises from a standard family of cycles) this presentation is aspherical (hence the cohomological dimension of G is at most 2). Moreover, the natural map of labelled graphs from Γ to the Cayley graph of G is a $(1/A, AL)$-quasi-isometry. The shortest relation of G is of length at least $Ag/2$, and any reduced word equal to e in G contains as a subword the reduction of at least one half of a word read on a cycle of Γ.

(Here $\|w\|$ is the length of the reduction of w; besides, in accordance with [GH90], by a (λ, c)-quasi-isometry we wean a map f such that $d(x,y)/\lambda - c \leqslant d(f(x), f(y)) \leqslant \lambda d(x,y) + c$.)

Remark 19. The same kind of theorem holds if we use the $C(7)$ condition instead of the $C'(1/6)$ condition, but in this case there is no control on the radius of injectivity (shortest relation length).

Remark 20. Using the techniques in [Del96] or [Oll04], the same kind of theorem should hold starting with any torsion-free hyperbolic group instead of the free group, provided that the girth of the graph is large enough w.r.t. the hyperbolicity constant, and that the labelling is quasi-geodesic. See [Oll].

Remark 21. Theorem 1 can be extended when the graph is infinite, in which case we get a direct limit of torsion-free, dimension-2 hyperbolic groups (but generally not hyperbolic), in which the conclusions of small cancellation theory still hold but with the isoperimetric constant for van Kampen diagrams tending to 0. In this case the small cancellation assumption reads: any piece has length less than 1/6 times the minimal length of a cycle on which it appears.

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