CUBULATING RANDOM GROUPS AT DENSITY LESS THAN 1/6

YANN OLLIVIER AND DANIEL T. WISE

Abstract. We prove that random groups at density less than $\frac{1}{6}$ act freely and cocompactly on CAT(0) cube complexes, and that random groups at density less than $\frac{1}{5}$ have codimension-1 subgroups. In particular, Property (T) fails to hold at density less than $\frac{1}{5}$.

Résumé. Nous prouvons que les groupes aléatoires en densité strictement inférieure à $\frac{1}{6}$ agissent librement et cocompactement sur un complexe cubique CAT(0). De plus en densité strictement inférieure à $\frac{1}{5}$, ils ont un sous-groupe de codimension 1 ; en particulier, la propriété (T) n'est pas vérifiée.

Introduction

Gromov introduced in [Gro93] the notion of a random finitely presented group on $m \geqslant 2$ generators at density $d \in (0;1)$. The idea is to fix a set $\{g_1, \ldots, g_m\}$ of generators and to consider presentations with $(2m-1)^{d\ell}$ relations each of which is a random reduced word of length ℓ (Definition 1.1). The *density* d is a measure of the size of the number of relations as compared to the total number of available relations. See Section 1 for precise definitions and basic properties, and [Oll05b, Gro93, Ghy04, Oll04] for a general discussion on random groups and the density model.

One of the striking facts Gromov proved is that a random finitely presented group is infinite, hyperbolic at density $< \frac{1}{2}$, and is trivial or $\{\pm 1\}$ at density $> \frac{1}{2}$, with probability tending to 1 as $\ell \to \infty$.

Żuk obtained Property (T) for a related class of presentations at density $> \frac{1}{3}$ (see [Żuk03] and the discussion in [Oll05b]). On the other hand, Gromov observed that at density $< d$, a random presentation satisfies the $C'(2d)$ small cancellation condition. Consequently, at density $< \frac{1}{12}$, the groups will not have Property (T) since $C'(\frac{1}{6})$ groups act properly discontinuously on CAT(0) cube complexes [Wis04].

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As above, the statements about the behavior of a group at a certain density are only correct with probability tending to 1 as $\ell \to \infty$. Throughout the paper, we will say that a given property holds *with overwhelming probability* if its probability tends exponentially to 1 as $\ell \to \infty$.

The goals of this paper are a complete geometrization theorem at $d < \frac{1}{6}$, implying the Haagerup property, and existence of a codimension-1 subgroup at $d < \frac{1}{5}$, implying [NR98] failure of Property (T):

Theorem 10.4. With overwhelming probability, random groups at density $d < \frac{1}{6}$ act freely and cocompactly on a CAT(0) cube complex.

Corollary 9.2. With overwhelming probability, random groups at density $d < \frac{1}{6}$ are a-T-menable (Haagerup property).

Theorem 7.4. With overwhelming probability, random groups G at density $d < \frac{1}{5}$ have a subgroup H which is free, quasiconvex and such that the relative number of ends $e(G, H)$ is at least 2.

Corollary 7.5. With overwhelming probability, random groups at density $d < \frac{1}{5}$ do not have Property (T).

CAT(0) cube complexes are a higher dimensional generalization of trees, which arise naturally in the splitting theory of groups with codimension-1 subgroups [Sag95, Sag97]. A group is *a-T-menable* or has the *Haagerup property* [CCJ$^+$01] if it admits a proper isometric action on a Hilbert space. This property is, in a certain sense, an opposite to Kazhdan's *Property* (T) [dlHV89, BdlHV08] which (for second countable, locally compact groups) is characterized by the requirement that every isometric action on an affine Hilbert space has a fixed point. There is also a definition of the Haagerup property in terms of a proper action on a space with measured walls [CMV04, CDH], which is a natural framework for some of our results.

The *relative number of ends* $e(G, H)$ of the subgroup H of the finitely generated group G is the number of ends of the Schreier coset graph $H\backslash G$ (see [Hou74, Sco78]). Note that $e(G, H)$ is independent of the choice of a finite generating set. We say H is a *codimension*-1 *subgroup* of G if H coarsely disconnects the Cayley graph Γ of G, in the sense that the complement $\Gamma - N_k(H)$ of some neighborhood of H contains at least two components that are not contained in any finite neighborhood $N_j(H)$ of H. The above two notions are very closely related and are sometimes confused in the literature: If $e(G, H) > 1$ then H is a codimension-1 subgroup of G, and the converse holds when there is more than one H-orbit of an "infinitely deep" component in $\Gamma - N_k(H)$.

Let us present the structure of the argument. In [Sag95], Sageev gave a fundamental construction which, from a codimension-1 subgroup H of G, produces an "essential" action of G on a CAT(0) cube complex. From [NR97] or [NR98] we know, in turn, that groups acting essentially/properly on a CAT(0) cube complex, act essentially/properly on a Hilbert space and cannot have Property (T) (their proof is a generalization of a proof in [BJS88] that infinite Coxeter groups are a-T-menable, which in turn, was a generalization of Serre's argument that an essential action on a tree determines an essential action on a Hilbert space [Ser80]).

In our situation the codimension-1 subgroups will arise as stabilizers of some codimension-1 subspaces, called *hypergraphs*, in the Cayley 2-complex $\widetilde{X}$ of the random group G. These hypergraphs are the same as those in [Wis04] and are defined in Section 2. The basic idea is, from the midpoint of each 1-cell in a 2-cell c, to draw a line to the midpoint of the opposite 1-edge in c (assuming all 2-cells have even boundary length). These lines draw a graph in the 2-complex, whose connected components are the hypergraphs. Hypergraphs are natural candidates to be walls [HP98].

In Section 4 we show that at density $d < \frac{1}{5}$, with overwhelming probability, the hypergraphs embed (quasi-isometrically) in the Cayley 2-complex. The main idea is that if a hypergraph self-intersects, it will circle around a disc in the Cayley 2-complex, thus producing a *collared diagram* (Section 3). But at $d < \frac{1}{5}$, the Dehn algorithm holds for a random group presentation [Oll07], so that in each van Kampen diagram some 2-cell has more than half its length on the boundary, which is impossible if a hypergraph runs around the boundary 2-cells of the diagram.

A consequence of this embedding property is that each hypergraph is a tree dividing $\widetilde{X}$ into two connected components, thus turning $\widetilde{X}$ into a space with walls [HP98].

We then show that these walls can be used to define (free quasiconvex) codimension-1 subgroups (Section 7). For this we need the complex $\tilde{X}$ to go "infinitely far away" on the two sides of a given wall. This is guaranteed by exhibiting a pair of infinite hypergraphs intersecting at only one point. At $d < \frac{1}{6}$, hypergraphs intersect at at most one point except for a degenerate case (Section 5). This is not true in general for $d < \frac{1}{5}$; however, one can still prove that through a "typical" 2-cell that a hypergraph Λ_1 passes through, there passes a second hypergraph Λ_2 transverse to Λ_1, which is enough (Section 6).

To prove the Haagerup property, we show that at $d < \frac{1}{6}$, the number of hypergraphs separating given points $p, q \in \tilde{X}$ is at least $\operatorname{dist}_{\tilde{X}}(p, q)/K$ for some constant K. Consequently the wall metric is quasi-isometric to the Cayley graph metric, which implies that the group has the Haagerup property. Key objects here are *hypergraph carriers* (the set of 2-cells through which a hypergraph travels): at $d < \frac{1}{6}$ these carriers are convex subcomplexes of $\tilde{X}$, but this is not the case at $d > \frac{1}{6}$. We were unable to prove that points are separated by a linear number of hypergraphs at $\frac{1}{6} \leqslant d < \frac{1}{5}$, where the failure of convexity substantially complicates matters, though we conjecture such a statement still holds.

Finally, Theorem 10.4 is proven by combining the various properties established at $d < \frac{1}{6}$ (including, most importantly, the separation of points by a linear number of hypergraphs) to see that the cubulation criteria in [HW04] are satisfied; these criteria guarantee that the action of G on the CAT(0) cube complex associated with a codimension-1 subgroup arising from hypergraphs is indeed free and cocompact.

At density $d > \frac{1}{5}$, our approachs completely fails: with overwhelming probability, there is only one hypergraph Λ, which passes through every 1-cell of the Cayley complex (Section 11). Its stabilizer is the entire group, and it is thus certainly not codimension-1. We do not know if there are codimension-1 subgroups at density $\frac{1}{5} < d < \frac{1}{3}$. But, as mentioned above, the transition at $d = \frac{1}{5}$ in the behavior of hypergraphs is related to another one, namely failure of the Dehn algorithm for $d > \frac{1}{5}$ [Oll07], and our intuition is that something of both combinatorial and geometric relevance really happens at $d = \frac{1}{5}$.

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1. Preliminaries and facts regarding Gromov's density

The density model of random groups was introduced by Gromov in [Gro93], Chapter 9 as a way to study properties of "typical" groups depending on the quantity of relators in a presentation of the group. We refer to [Oll05b, Gro93, Ghy04, Oll04] for general discussions on random groups and the density model.

Definition 1.1 (Density model of random groups)**.** Let $m \geqslant 2$ be an integer and consider the free group F_m generated by $a_1^{\pm 1}, \ldots, a_m^{\pm 1}$.

Let $0 \leqslant d \leqslant 1$ be a density parameter. Let ℓ be a (large) length. Choose $(2m-1)^{d\ell}$ times (rounded to the nearest integer) at random a reduced word of length ℓ in the letters $a_1^{\pm 1}, \ldots, a_m^{\pm 1}$, uniformly among all such words. Let R be the set of words so obtained.

A *random group at density* d *and length* ℓ is the group $G = F_m/\langle R\rangle$, whose presentation is $\langle\, a_1, \ldots, a_m \mid R\,\rangle$.

A property is said to occur *with overwhelming probability* in this model, if its probability of occurrence tends exponentially to 1 as $\ell \to \infty$.

Note that a priori, repetitions are allowed in the choice of the random words (so that the choices are independent); but actually when $d < 1/2$, with overwhelming probability there are no repetitions.

The basic intuition is that at density d, subwords of length $(d-\varepsilon)\ell$ of the relators in the presentation will exhaust all reduced words of length $(d-\varepsilon)\ell$.

The interest of the model is established through the following sharp phase transition theorem, proven by Gromov [Gro93] (see also [Oll04]):

Theorem 1.2 (M. Gromov)**.** *Let $d < 1/2$. Then with overwhelming probability, a random group at density d is infinite, hyperbolic, torsion-free.*

Let $d > 1/2$. Then with overwhelming probability, a random group at density d is either $\{1\}$ or $\{1, -1\}$.

One of the motivations for the results in this paper is the following ([Żuk03], see also the discussion in [Oll05b]):

Theorem 1.3 (A. Żuk)**.** *Let $d > 1/3$. Then with overwhelming probability, a random group at density d has Property (T).*

It is not known whether 1/3 is optimal in this theorem. Our results imply that 1/5 is a lower bound.

Remark 1.4. According to the definition above, all relators in a random group have exactly the same length. However, the results stay the same if we take relators of length between ℓ and $\ell + C$ where C is any constant independent of ℓ.

Some results on random groups, including Theorem 1.2 and Theorem 1.3, also extend to the case when relators are taken of length between ℓ and $C\ell$ for some $C > 1$ (see [Oll04]), but we do not know if this is the case for the main theorems presented in this paper.

Hyperbolicity of random groups at $d < 1/2$ is proven using isoperimetry of van Kampen diagrams. In this paper we shall repeatedly need a precise statement of this isoperimetric inequality, which we state now.

For a van Kampen diagram D, we use the notation $|\partial D|$ for the length of its boundary path, and the notation $|D|$ for the number of 2-cells in D.

Convention 1.5. When a property of a random group depends on a parameter ε, the phrase "the property occurs with overwhelming probability" will mean that for any $\varepsilon > 0$, the probability of the property tends exponentially to 1 as $\ell \to \infty$. (This may not be uniform in ε.)

The following, proven in [Oll07], is a strengthening of the original statement of Gromov, which held only for diagrams of size bounded by some constant. Note the role of $d = \frac{1}{2}$.

Theorem 1.6. *At density d, for any $\varepsilon > 0$ the following property occurs with overwhelming probability: all reduced van Kampen diagrams D satisfy*

$$|\partial D| \geqslant (1 - 2d - \varepsilon)\ell \, |D|$$

When using this result in this paper we will often omit the ε.

We now gather some definitions pertaining to small cancellation. We refer to chapter V of [LS77] for the definition of a *piece* in a group presentation.

Definition 1.7 (Small cancellation)**.** A presentation satisfies the $C'(\alpha)$ *condition*, with $0 \leqslant \alpha \leqslant 1$, if for each relator R, and each piece P occurring in R, we have $|P| < \alpha \, |R|$.

A presentation satisfies the $B(2p)$ *condition* if every word w which is a concatenation of at most p pieces and which is a subword of a relator R satisfies $|w| \leqslant \frac{1}{2} \, |R|$.

A presentation satisfies the $C(p)$ *condition* if no relator R is the concatenation of fewer than p pieces.

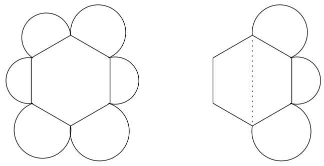

FIGURE 1. Diagrams contradicting the $C(7)$ and $B(6)$ conditions.

Note that $C'(\frac{1}{2p}) \Rightarrow B(2p) \Rightarrow C(2p)$ but that none of the reverse implications hold.

Proposition 1.8. *With overwhelming probability:*
1. *The $C'(\alpha)$ condition occurs at density $< \alpha/2$.*
2. *The $B(6)$ condition occurs at density $< \frac{1}{8}$.*
3. *The $C(p)$ condition occurs at density $< \frac{1}{p}$.*

Proof. The proof for $C'(\alpha)$ is written in detail in [Gro93], § 9.B. Let us briefly recall the argument. Since the number of reduced words of length L is $(2m)(2m-1)^{L-1}$, the probability that two random reduced words of length ℓ share a common initial subword of length $L \leqslant \ell$ is $\left((2m)(2m-1)^{L-1}\right)^{-1} \leqslant (2m-1)^{-L}$.

So given two random words of length ℓ, the probability that they share a piece of length L is less than $\ell^2(2m-1)^{-L}$ where the ℓ^2 accounts for the choice of the position at which the piece occurs.

Now in a random group at density d, there are by definition $(2m-1)^{d\ell}$ relators. So the probability that there exists a couple of relators in the presentation having a piece of length L is at most $\ell^2(2m-1)^{2d\ell}(2m-1)^{-L}$ since there are $(2m-1)^{2d\ell}$ possible choices of couples of relators (we also have to check the special case when a relator shares a piece with itself, but this is not difficult). So if $L = \alpha\ell$ this makes $\ell^2(2m-1)^{(2d-\alpha)\ell}$. If $d < \alpha/2$, this tends to 0 as $\ell \to \infty$ (but all the more slowly as d is close to $\alpha/2$). One can reverse the argument to see that if $d > \alpha/2$, such an event actually occurs.

It is worth to compare this with Theorem 1.6. Indeed, when two relators share a piece of length $\alpha\ell$ we can form a van Kampen diagram D of boundary length $|\partial D| = 2\ell - 2\alpha\ell = |D|\,\ell(1-\alpha)$ so that this diagram contradicts Theorem 1.6 when $d < \alpha/2$.

The $C(p)$ condition amounts to the exclusion of a reduced van Kampen diagram D in which a 2-cell is surrounded by at most $p-1$ 2-cells as on the left of Figure 1. Such a diagram D satisfies $|\partial D| \leqslant p\ell - 2\ell$ whereas Theorem 1.6 yields $|\partial D| \geqslant p\ell(1-2d-\varepsilon)$ so that (choosing $\varepsilon = (1/p-d)/10$) this is a contradiction when $d < 1/p$. This proves statement (3).

The $B(6)$ condition amounts to the exclusion of a diagram in which half the boundary of a 2-cell is covered by three other 2-cells as on the right of Figure 1. Note that this diagram D satisfies $|\partial D| = 4\ell - 2(\ell/2) = 3\ell$.

Theorem 1.6 implies that $|\partial D| \geqslant 4\ell(1-2d-\varepsilon)$, so $d \geqslant 1/8 - \varepsilon/2$. So if $d < 1/8$ we get a contradiction (choosing e.g. $\varepsilon = (1/8-d)/10$). □

Remark 1.9. By [Wis04], hyperbolicity and the $B(6)$ condition together imply the existence of a free and cocompact action on a CAT(0) cube complex. So this conclusion holds at density $< \frac{1}{8}$. This is a bit stronger than the $< \frac{1}{12}$ condition mentioned in the introduction.

Since the $C(6)$ condition is satisfied at density $< \frac{1}{6}$, our results suggest that generic $C(6)$ groups are a-T-menable. It is currently an open problem whether or not every infinite $C(6)$ group fails to satisfy Property (T).

In this paper we shall sometimes need to avoid some annoying topological configuration. This is the object of the next two propositions.

Proposition 1.10. *Let G be a random group at density $d < 1/4$. Let p be a closed path embedded in the Cayley graph of G. Then the length of p is at least ℓ; moreover, either p is the boundary path of some relator in the presentation, or the length of p is at least $\ell + \ell(1-4d-\varepsilon)$.*

Consequently, the boundary paths of relators embed.

Proof. This results from Theorem 1.6. Indeed, since p is not homotopic to 0, it is the boundary path of some van Kampen diagram D with at least one 2-cell, and so $|p| \geqslant \ell\,|D|\,(1-2d-\varepsilon)$. Now either $|D| = 1$ and p is the boundary path of a relator, or $|D| \geqslant 2$ and $|p| \geqslant 2\ell(1-2d-\varepsilon)$. □

Corollary 1.11. *Let G be a random group at density $d < 1/4$ and let $\widetilde{X}$ be the Cayley complex associated to the presentation. Let c_1, c_2 be two 2-cells in $\widetilde{X}$. Then $\partial c_1 \cap \partial c_2$ is connected.*

Proof. Suppose not and let v, w be two 0-cells of $\widetilde{X}$ lying in different components of $\partial c_1 \cap \partial c_2$. Let p_1, p_1' be the two paths in ∂c_1 joining v to w on each side of c_1, and likewise let p_2, p_2' be the two paths in ∂c_2 joining v to w.

Each of the paths $p_1 p_2^{-1}, p_1 {p_2'}^{-1}, p_1' p_2^{-1}$ and $p_1' {p_2'}^{-1}$ is a closed path in the 1-skeleton of $\tilde{X}$. Each of these paths is not null-homotopic in this 1-skeleton, otherwise v and w would lie in the same component of $\partial c_1 \cap \partial c_2$. So by Proposition 1.10 each of these paths has length at least ℓ, and since $|p_1| + |p_1'| = |p_2| + |p_2'| = \ell$, the only possibility is that $|p_1| = |p_1'| = |p_2| = |p_2'| = \ell/2$. This implies that $\left|p_1 p_2^{-1}\right| = \ell$, so that $p_1 p_2^{-1}$ is the boundary path of some 2-cell c_3. Now c_1 and c_3 share half of their boundary length, which at $d < 1/4$ contradicts Proposition 1.8. □

Another notion we shall need is that of *fulfilling* of a diagram. Let D, an *abstract diagram*, be a finite connected graph embedded in the plane, each edge of which is decorated with a positive integer, its *length.* Let $\langle\, a_1, \ldots, a_m \mid R \,\rangle$ be any group presentation. A *fulfilling* of D is the attribution to each face of D of a relator in R (together with an orientation) such that the resulting object is a reduced van Kampen diagram of the presentation, in a way compatible with the prescribed lengths (see the notion of *decorated abstract van Kampen diagram* in [Oll04] for precisions).

The following appears in [Oll05a], Propositions 12 and 13:

Theorem 1.12. *Let $G = \langle a_1, \ldots, a_m \mid R \rangle$ be a group presentation. For any abstract diagram D, let $S_n(D)$ be the number of n-tuples of distinct relators in R such that there exists a fulfilling of D using these relators (n is at most the number of faces $|D|$ of D since a relator may be used multiple times in the diagram).*

For random groups at density d, for any abstract diagram D we have the following bound on the expectation of $S_n(D)$:

$$\mathbb{E}S_n(D) \leqslant (2m-1)^{\frac{1}{2}(|\partial D|-(1-2d)\ell|D|)}$$

and so for any D, with overwhelming probability we have:

$$S_n(D) \leqslant (2m-1)^{\frac{1}{2}(|\partial D|-(1-2d-\varepsilon)\ell|D|)}$$

We note that the second assertion in Theorem 1.12 (which holds for fixed D) follows from the first one by the Markov inequality.

2. Hypergraphs and carriers

2.1. **Historical background on cubulating groups.** The results in this paper employ Sageev's construction [Sag95] of an action on a CAT(0) cube complex from a group G and a codimension-1 subgroup H.

Niblo and Reeves [NR03] and Wise [Wis04] had observed that Sageev's construction works in the context of "geometric spaces with walls". For Coxeter groups, these walls are the reflection walls stabilized by the involutions in the Coxeter complex. For small cancellation groups, the walls are constructed as we do here: by producing immersed graphs in a 2-complex that are transverse to the 1-skeleton and such that each edge of the graph bisects a 2-cell. The walls corresponding to such graphs appear to have played a role in Ballmann-Świątkowski's proof of the failure of Property (T) for the geometric case of (4, 4)-complexes and (6, 3)-complexes [BŚ97].

It is clear from [Wis04] that Sageev's cubulation result can be carried out for a family of more general codimension-1 graphs which embed, are transverse to the 1-skeleton, and locally separate the 2-complex. These are examples of Dunwoody's "tracks" and we expect they will be referred to as "walls" in future work on this subject. Indeed, subsequently, Nica [Nic04] and Chatterji and Niblo [CN05] have written out an explicit application of Sageev's construction to cubulate abstract "spaces with walls". Those were introduced by Haglund and Paulin [HP98] especially motivated by Coxeter groups and CAT(0) cube complexes.

Building upon [Sag97, NR03, Wis04], Hruska and Wise [HW04] have laid out "axioms" on a space with walls (or 2-complex with hypergraphs) for verifying finiteness properties of the cubulation. We follow the framework there to verify our main results. We expect there will be further work along these lines.

2.2. **Definition of hypergraphs.** In a nutshell, hypergraphs in a 2-complex are obtained by drawing a segment between the midpoints of each pair of opposite 1-cells in each 2-cell. These segments define a graph, the connected components of which are the hypergraphs. We give a more precise definition below.

Definition 2.1. Let $\widetilde{X}$ be a simply connected 2-complex. We suppose that each 2-cell of $\widetilde{X}$ has even boundary length. (If this is not the case, we just perform a subdivision of all 1-edges of $\widetilde{X}$ before constructing hypergraphs.)

We define a graph Γ as follows: The set of vertices of Γ is the set of 1-cells of $\widetilde{X}$. There is an edge in Γ between two vertices if there is some 2-cell R of $\widetilde{X}$ such that these vertices correspond to antipodal 1-cells in the boundary of R (if there are several such 2-cells R, we put as many edges in Γ). The 2-cell R is the 2-cell of $\widetilde{X}$ *containing* the edge.

There is a natural map φ from Γ to a geometric realization of it in $\widetilde{X}$, which sends each vertex of Γ to the midpoint of the corresponding 1-cell of $\widetilde{X}$, and each edge of Γ to a segment joining two antipodal points in the 2-cell R. Note that the images of two edges contained in the same 2-cell R always intersect, so that in general φ is not an embedding.

A *hypergraph* in $\widetilde{X}$ is a connected component of Γ. The 1-cells of $\widetilde{X}$ through which a hypergraph passes are *dual* to it. The hypergraph Λ *embeds* if φ is an embedding from Λ to its geometric realization in $\widetilde{X}$, i.e. if no two distinct edges of Λ live in the same 2-cell of $\widetilde{X}$.

For each subgraph $A \subset \widetilde{\Gamma}$, we define a 2-complex V, the *unfolded carrier* of A, in the following way: For each edge e in A contained in the 2-cell R of $\widetilde{X}$, consider an isomorphic copy R_e of R. Now take the disjoint union of these copies and glue them as follows: if edges e and e' of A share a common endpoint $v \in A$, identify R_e and $R_{e'}$ along the 1-cell corresponding to vertex v. When A is connected, it is by construction an embedded hypergraph of its unfolded carrier.

A *hypergraph segment* (resp. *ray*, resp. *line*) is an immersed finite path (resp. immersed ray, immersed line) in a hypergraph. A *ladder* is the unfolded carrier of a segment.

Remark 2.2. The term "hypergraph" is a misnomer, which arose as a graph corresponding to a "hyperplane" in a CAT(0) cube complex C. Hypergraphs will play the role of "codimension-1 subgraphs" later in the paper. The term hypergraph is used in graph theory to mean a certain high-dimensional generalization of a graph, but we will have no use for that notion in this paper.

Lemma 2.3. *Suppose a hypergraph Λ is an embedded tree in the simply connected complex $\widetilde{X}$. Then $\widetilde{X} - \Lambda$ consists of two components.*

Proof. This follows easily from the fact that $\mathsf{H}_1(\widetilde{X}) = 0$ and a Mayer-Vietoris sequence argument applied to the complement of the hypergraph and a neighborhood of the hypergraph. □

3. Studying hypergraphs with collared diagrams

In this section we define and examine various notions of "collared diagrams". In Section 3.1, we show that hypergraphs are trees unless certain collared diagrams exist. In Section 3.3, we show that the intersection of a pair of hypergraphs contains at most one point, unless there is a certain collared diagram between them. In Section 3.4, we explain that if a geodesic touches a hypergraph in exactly two points, then there is a certain relatively collared diagram between the geodesic and the hypergraph. In each case, various quasicollared diagrams will serve as a useful technical object to facilitate the proofs.

Convention 3.1 (Conventions on $\widetilde{X}$ and its hypergraphs)**.** In the remainder of the paper, $\widetilde{X}$ is the Cayley 2-complex of a random group (and hence all relations have the same length). However in this section we work under more general hypotheses (which the reader is welcome to ignore). Our only hypothesis on $\widetilde{X}$ is that it is a simply connected combinatorial 2-complex, and that the boundary cycle of each 2-cell is an immersed path in $\widetilde{X}^1$, of even length.

3.1. **Collared diagrams.**

We refer to [MW02] (Def. 2.6) for the definition of *disc diagrams*, which play for arbitrary 2-complexes the role of van Kampen diagrams for Cayley complexes. The reader may just read "van Kampen diagram".

The central notion in this section is the following (see Figure 2):

Definition 3.2 (Collared diagram)**.** A *collared diagram* is a disc diagram $D \to \widetilde{X}$ with the following properties:

(1) there is an external 2-cell C called a *corner* of D
(2) there is a hypergraph segment $\lambda \to D \to \widetilde{X}$ of length at least 2
(3) the first and last edge of λ lie in C, and no other edge lies in C
(4) λ passes through every other external 2-cell of D exactly once
(5) λ does not pass through any internal 2-cell of D.

D is *cornerless* if moreover the first and last edge of λ coincide in C (in which case the hypergraph cycles).

The above definition implies that the diagram is homeomorphic to a disc.

Remark 3.3. Note that we do not exclude that C is the only boundary 2-cell of D (in which case the boundary path of C is not simple). However, since the boundary path of any 2-cell is immersed by assumption, D has at least two 2-cells. But it might not have any internal 2-cells.

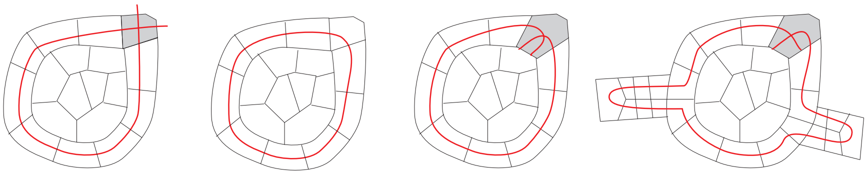

FIGURE 2. Several kinds of collared diagrams: The corner 2-cell of the first collared diagram on is shaded. The second collared diagram is cornerless. The hypergraph segment of the third collared diagram ends before it enters the interior. The last collared diagram is more typical in the sense that the carrier of the segment folds. (Note that the figure is topological only, so that some 1-cells may appear longer or shorter than others, yet hypergraphs still cross 2-cells at antipodal pairs of 1-cells.)

Definition 3.4. A *cancellable pair* in $Y \to X$ is a pair of distinct 2-cells R_1, R_2 meeting along an edge e in Y such that R_1 and R_2 map to the same 2-cell in X, and moreover, the boundary paths of R_1 and R_2 starting at e, map to the same path in X. A map $Y \to X$ is *reduced* if Y contains no cancellable pairs. Note that the composition of reduced maps is reduced.

In our framework, $Y \to X$ is reduced precisely if $Y \to X$ is a *near-immersion* meaning that $(Y - Y^0) \to X$ is an immersion. See [MW02] for more about reduced maps.

For van Kampen diagrams this notion coincides with the usual notion of reduced diagram (at least if relators which are proper powers are handled correctly, which is a messy point in the van Kampen diagram literature).

The main goal of this section is to prove the following theorem.

Theorem 3.5. *Let Λ be some hypergraph. The following are equivalent:*

(1) *Λ is an embedded tree.*
(2) *There is no reduced collared diagram collared by a segment of Λ.*
(3) *There is no quasicollared diagram collared by a segment of Λ (Definition 3.6 below).*

Proof. If there is a reduced diagram $E \to \widetilde{X}$ collared by a segment λ of Λ then clearly either Λ is not a tree or $\Lambda \to \widetilde{X}$ is not an embedding. Indeed, the path $\lambda \to E$ has the property that its first and last edges cross or coincide, and so this is the case for $\lambda \to \widetilde{X}$.

The converse, which plays an important role in this paper needs a bit more work, and so we outline the proof which employs several lemmas proven later in this section. Suppose Λ is not an embedded tree.

In Lemma 3.8, we prove that if Λ is not an embedded tree in $\widetilde{X}$, then there exists a diagram quasicollared by Λ (Definition 3.6), denoted by $F \to \widetilde{X}$.

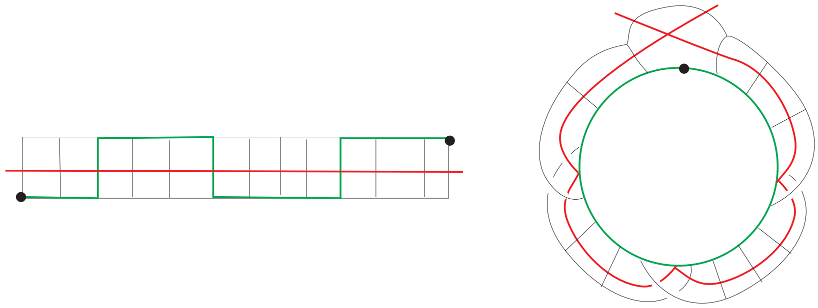

FIGURE 3. On the left is the path P in the ladder L containing part of the hypergraph Λ. On the right is the quasicollared disc diagram $L \cup_P D$ obtained by attaching L to D along P.

In Lemma 3.9, we show that by removing cancellable pairs, we can assume that $F \to \widetilde{X}$ is reduced.

In Lemma 3.10 we extract a reduced collared diagram $E \to \widetilde{X}$ from the reduced quasicollared diagram $F \to \widetilde{X}$. □

We now define quasicollared diagrams which, unlike collared diagrams, do not have an easily stated intrinsic definition.

Definition 3.6 (Quasicollared diagram)**.** Consider the ladder L of some hypergraph segment λ of length at least 2. We suppose that the first and last 2-cells C_1, C_2 of L map to the same two-cell of $\widetilde{X}$.

Let $A = L/\{C_1{=}C_2\}$ be the complex obtained from L by identifying the closures of C_1 and C_2 (using the maps $C_1 \to \widetilde{X}$, $C_2 \to \widetilde{X}$).

Let $P \to A$ be a simple cycle in A representing a generator of $\mathsf{H}_1(A)$. Suppose that there exists a disc diagram $D \to \widetilde{X}$ with boundary path P.

A *quasicollared diagram* $F \to X$ is the complex obtained by forming the union $F = A \cup_P D$. (See Figure 3.) We say that F is *collared by* the hypergraph segment λ.

Remark 3.7. F is a genuine disc diagram precisely when $P \to A$ does not cross λ. This happens precisely when A is a cylinder instead of a Moebius strip and $P \to A$ is a boundary cycle of A.

Lemma 3.8 (Existence)**.** *Suppose that the hypergraph Λ is not an embedded tree in $\widetilde{X}$. Then there exists a quasicollared diagram $F \to \widetilde{X}$ that is collared by Λ.*

Proof. The hypergraph Λ is not an embedded tree if and only if there exists a nontrivial immersed edge-path $\lambda \to \Lambda$ such that λ projects to a non-simple path in $\widetilde{X}$. We can assume that the hypergraph segment λ is minimal, that is, any proper subsegment of λ embeds.

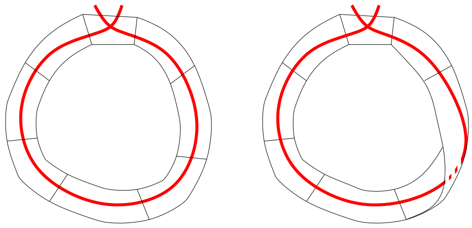

FIGURE 4. The basic loops.

So the 2-cells of $\widetilde{X}$ containing the first and last edge of λ are the same. Let L be the ladder carrying λ; its first and last 2-cells C_1, C_2 map to the same 2-cell of $\widetilde{X}$.

We can therefore form a quotient space $A = L/\{C_1{=}C_2\}$ and there is an induced map $A \to \widetilde{X}$. We will refer to the cell $C_1 = C_2$ as the *corner*. As in Figure 4, there are two cases for A according to whether or not λ "preserves orientation" of A.

Let $P \to A$ be a simple cycle in A that maps to a generator of $\pi_1(A)$. Since $\widetilde{X}$ is simply connected, there is a disc diagram $D \to \widetilde{X}$ whose boundary path is P.

Note that while $P \to A$ is an immersion, the map $A \to \widetilde{X}$ may not be, and so it is possible that D is singular, and may have spurs.

Finally we form the desired quasicollared diagram $F = A \cup_P D$.

If we think of P as a path in L instead of A, then P may travel from one side of L to the other, as in Figure 3. Indeed, this is always the case in the orientation reversing case where A is a Moebius strip. F is a genuine disc diagram exactly when λ does not cross any edge of P. □

Lemma 3.9 (Reducing)**.** *Let $F \to X$ be a quasicollared diagram. Then there exists a reduced quasicollared diagram $F' \to X$ which is collared by a subsegment of the hypergraph segment collaring F.*

Proof. Keeping the notation in the definition of quasicollared diagrams, there are three types of cancellable pairs in $F \to \widetilde{X}$ to consider according to whether the 2-cells lie in: D, D or D, A, or A, A.

In the first case, the cancellable pair is removed in the usual way for van Kampen diagrams (prone to errors in the literature, but works nevertheless...): We remove the open 2-cells and the open 1-cell along which they form a cancellable pair, and we identify their remaining corresponding boundaries.

In the second case, we can adjust our choice of P to form a new simple cycle. Namely let R_1, R_2 be the 2-cells forming the cancellable pair, with $R_1 \subset A$ and $R_2 \subset D$. Push P across to the other side of R_1. Now R_2 can be removed from D. This is illustrated as the first two configurations in Figure 5, where cancellable pairs are marked by dots: the first configuration is the case when originally P does not jump from one side of A to the other

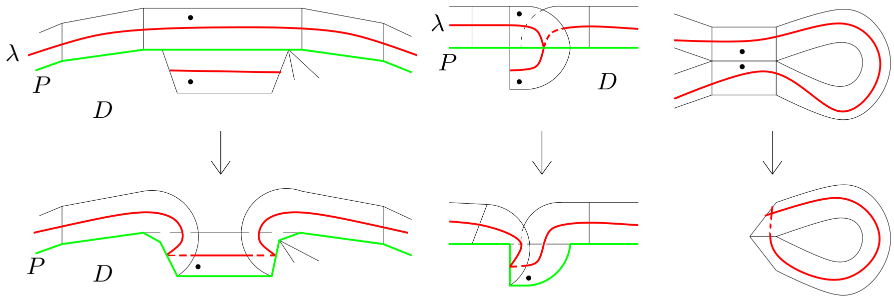

FIGURE 5. A cancellable pair between D and A, another cancellable pair in a slightly different position, and a cancellable pair between A and A. The cancellable pairs are marked with dots.

around some side of R_1 (in which case the new P crosses A twice at this point); the second configuration is when originally P crosses A along some side of R_1, in which case the new P crosses A along the other side of R_1 afterwards.

In the third case (reduction between A and A), this implies that there is a pair of 2-cells R_1, R_2 in L, different from the pair of extremal 2-cells, mapping to the same 2-cell of $\widetilde{X}$. This means that we can find a proper subsegment λ' of the hypergraph segment λ which does not embed in $\widetilde{X}$. The ladder carrying λ', which has R_1 and R_2 as extremal 2-cells, can now be used to define a smaller quasicollared diagram as in the rightmost illustration of Figure 5.

Keep reducing cancellable pairs. The only thing to check is that eventually A is not empty. Observe that reductions between D and D and between D and A preserve A and λ. So the only way A can become empty is if at some step two *consecutive* 2-cells of A are cancellable. But this means that λ was not immersed, which it is by definition of a hypergraph segment. □

We now extract a collared diagram from a quasicollared one.

Lemma 3.10 (Collaring)**.** *If there is a reduced quasicollared diagram F, then there is a reduced collared diagram F'. Moreover, F' and F are collared by segments of the same hypergraph of $\widetilde{X}$.*

Proof. Keeping the same notation again, suppose some edge e of P crosses Λ, that is, consider an edge e in P that is dual to Λ (witnessing for the fact that the diagram is quasicollared but not collared). Observe that Λ enters D at e. Let λ' be the path of Λ in D issuing from e. As on the left in Figure 6, either λ' is simple or λ' crosses itself in D.

If λ' crosses itself then we choose some subpath λ'' of λ' that is a simple loop in D bounding some topological disc in D, as in the middle diagram of Figure 6.

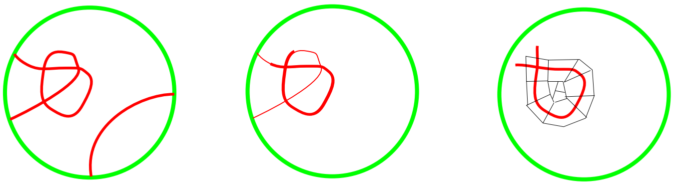

FIGURE 6.

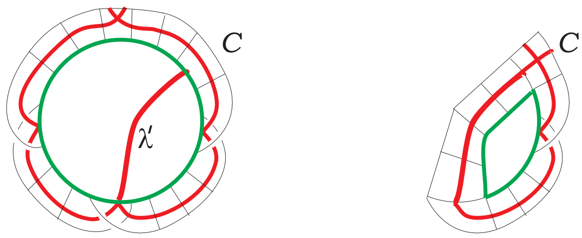

FIGURE 7.

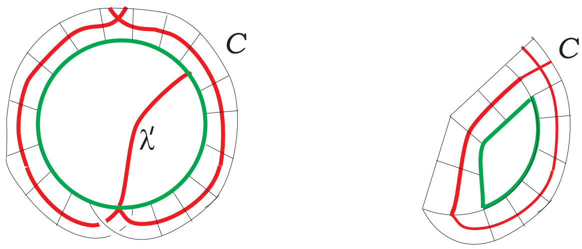

FIGURE 8.

There is then a diagram D' having λ'' as the hypergraph in its collar. This is illustrated on the right in Figure 6. (Note that D' is only nearly a subdiagram of D since the map $D' \to D$ might fail to be injective on $\partial D'$.)

The other possibility is that the path λ' is simple in D (Figure 7). In this case, λ' has to exit D by crossing the collar at some 2-cell C, dividing F into two halves. Pick the half of F that does not contain the corner of F: this provides a new quasicollared diagram F' with C as its corner. (If C happens to be the corner of F already, then any half will do.)

This new diagram is smaller than F in the sense that the number of intersections between ∂D and the hypergraph segment collaring the diagram decreases. So repeating the process will eventually provide a collared diagram. The last step is illustrated on Figure 8.

The new diagram obtained is reduced since F itself is. □

3.2. **Diagrams quasicollared by hypergraphs and paths.** We now give a definition of a notion generalizing that of quasicollared diagram, in which

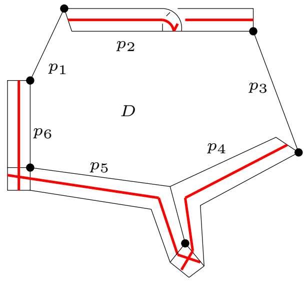

FIGURE 9. A diagram collared by hypergraphs and paths.

we allow the collar to consist of segments of several hypergraphs and/or paths in $\widetilde{X}$.

Definition 3.11. Let $n \geqslant 2$ be an integer and decompose $\{1, \dots, n\}$ as a disjoint union $I \cup J$ (where I or J may be empty). For $i \in I$ let λ_i be a hypergraph segment of length at least 2, carried in ladder L_i. For $i \in I$ let also p_i be a path immersed in L_i joining a point in the boundary of the first 2-cell of L_i and a point in the boundary of the last 2-cell of L_i. For $j \in J$ let p_j be any path immersed in $\widetilde{X}$.

We suppose that (subscripts mod n):

(1) When $i \in I$ and $i+1 \in I$, then: The final 2-cell of L_i and the initial 2-cell of L_{i+1} have the same image in $\widetilde{X}$. Moreover the final edge of λ_i and the initial edge of λ_{i+1} do not have the same image in $\widetilde{X}$. Moreover the (images in $\widetilde{X}$ of) initial point of p_{i+1} and the final point of p_i coincide.
(2) When $i \in I$ and $i+1 \in J$, then the image in $\widetilde{X}$ of the final point of p_i coincides with the initial point of p_{i+1}, and likewise when $i \in J$ and $i+1 \in I$.
(3) If $i \in J$ then both $i+1$ and $i-1$ lie in I.

This allows to define a cyclic path $P = \cup p_i$. Let D be a disc diagram with boundary path P.

Let A' be the disjoint union of L_i for $i \in I$ and let A be the quotient of A' under the identification of the last 2-cell of L_i with the first 2-cell of L_{i+1} whenever $i, i+1 \in I$.

A *diagram quasicollared by the* $\lambda_i, i \in I$ *and the* $p_j, j \in J$ is the union $E = D \cup_{p_i, i \in I} A$.

The *corners* of E are the initial and final 2-cells of the L_i's.

It is said to be *collared by the* $\lambda_i, i \in I$ *and the* $p_j, j \in J$ if E is a genuine disc diagram.

We say that the hypergraphs λ_i *do not enter* E if for $i \in I$, the initial and final points of λ_i belong to the boundary of E.

Note that D may be singular, since the map from the unfolded carrier of a hypergraph to $\widetilde{X}$ generally identifies a lot of 1-cells and this will result in "spurs" in D. D may even contain no 2-cell in the case P is a null-homotopic path in the 1-skeleton of $\widetilde{X}$. Note also that we do not allow "cornerless" diagrams since we imposed that two successive hypergraph segments intersect transversely.

3.3. **2-collared diagrams.** A 2-*collared diagram* is a diagram collared by two hypergraph segments.

The main goal of this section is the following:

Theorem 3.12. *Suppose Λ_1 and Λ_2 are distinct hypergraphs that are embedded trees in $\widetilde{X}$. There is more than one point in $\Lambda_1 \cap \Lambda_2$ if and only if there exists a reduced diagram E collared by segments of Λ_1 and Λ_2. Moreover, if Λ_1 and Λ_2 cross at a 2-cell C, then we can choose E so that C is one of its corners.*

Proof. If there exists a diagram $E \to \widetilde{X}$ collared by Λ_1 and Λ_2, then the intersections of λ_1 and λ_2 in the two corners 2-cells map to two intersection points in $\widetilde{X}$, which are distinct since Λ_1 and Λ_2 are embedded trees.

The converse requires more work, and we outline its proof which depends upon lemmas proven in this section.

In Lemma 3.13, we show that if Λ_1 and Λ_2 intersect twice, then there is a quasicollared diagram between them.

In Lemma 3.15, we show that if there is a quasicollared diagram between them then there is a reduced quasicollared diagram between them.

In Lemma 3.16, we extract a reduced collared diagram between Λ_1 and Λ_2, from a reduced quasicollared diagram. □

Lemma 3.13 (Existence of 2-quasicollared diagrams)**.** *Suppose there are distinct hypergraphs Λ_1 and Λ_2 which are embedded trees and whose images in $\widetilde{X}$ intersect in more than one point. Then there exists a diagram F quasicollared by Λ_1 and Λ_2; moreover, its corners can be taken to be an arbitrary pair of distinct 2-cells where Λ_1 and Λ_2 intersect.*

Proof. Let λ_1, λ_2 be hypergraph segments in Λ_1, Λ_2 which intersect at the centers of their first and last edges. Let L_i be the ladder carrying λ_i, and observe that the first and last 2-cells of L_1, L_2 project to the same 2-cells of $\widetilde{X}$. Let $A \to \widetilde{X}$ be obtained by forming the union of L_1 and L_2 and identifying their first and last closed 2-cells. Observe that $\pi_1(A) \cong \mathbb{Z}$ except for the degenerate case where each L_i consists entirely of these first and last 2-cells.

In this degenerate case, define $F = A$. Otherwise, let $P \to A$ be a simple closed path representing a generator of $\pi_1(A)$. Let $D \to \widetilde{X}$ be a disc diagram with boundary path $P \to \widetilde{X}$. Let $F = A \cup_P D$. □

Lemma 3.14. *Let F be a diagram quasicollared by two embedded hypergraph segments λ_1, λ_2. Then there exists a diagram F' quasicollared by two subsegments λ_1', λ_2' of λ_1, λ_2, such that the only 2-cells in the intersection of the* *images of the ladders of λ'_1 and λ'_2 in $\widetilde{X}$ are the corners of F'. Moreover, F' can be chosen to contain either corner of F as one of its corners.*

Proof. This is more difficult to state than to prove. Let C_1 be the first corner of F. Let C_2 be the first 2-cell in the ladder of λ_1, distinct from C_1, which lies in the image of the ladder of λ_2 in $\widetilde{X}$. Taking the corresponding initial subsegments of λ_1 and λ_2 and applying Lemma 3.13, we get what we need, preserving corner C_1. □

Lemma 3.15 (Reducing)**.** *Let F be a diagram quasicollared by two hypergraphs which are embedded trees. Then there exists a reduced diagram F' quasicollared by two segments of the same hypergraphs, and moreover F' can be chosen to contain either corner of F.*

Proof. This is similar to Lemma 3.9. Applying Lemma 3.14, we can suppose that the images of the ladders collaring F intersect only at the two corners of F.

Keep the notation of Definition 3.11. Cancellable pairs between 2-cells both in D can be removed as usual to lower the total number of 2-cells.

Now for the case of cancellable pairs between A and A. Since each hypergraph is an embedded tree, the two cells of the pair cannot lie in the same ladder L_i. But the cancellation cannot occur between L_1 and L_2 either since this would contradict the conclusion of Lemma 3.14.

Finally, if there is a cancellable pair between D and A, then we can push the boundary path P across the 2-cell in A (compare Lemma 3.9), obtaining a new closed path P', which is the boundary path of a disc diagram D' with one less 2-cell. (Note that this operation preserves A.) □

Lemma 3.16 (Collaring)**.** *Let F be a reduced diagram quasicollared by two hypergraphs Λ_1, Λ_2 which are embedded trees. Then there is a reduced diagram F' collared by Λ_1 and Λ_2. Moreover, F' can be chosen to share either corner with F.*

Proof. Keeping notation again, note that if the path P does not intersect λ_1 and λ_2 then the diagram is collared.

Now suppose that P crosses, say, the segment λ_1. (The argument for λ_2 is identical.) Consider the first such situation on λ_1. Then λ_1 can be extended into a hypergraph segment μ_1 that enters D. Since Λ_1 is an embedded tree, μ_1 cannot cross itself. So μ_1 exits D by crossing λ_2, not λ_1. (This contrasts with Lemma 3.10, where we did not assume that the hypergraph was an embedded tree.) Then by choosing the part of F lying between μ_1 and λ_2 we get a diagram which is quasicollared by λ_2 and μ_1, containing the first corner of F. Repeating the argument with λ_2 produces a diagram collared by μ_1 and a segment μ_2 of Λ_2, which ends the proof. □

3.4. Diagrams collared by a hypergraph and a path.

Lemma 3.17 (Existence)**.** *Let Λ be a hypergraph which is an embedded tree in $\widetilde{X}$. Let λ be a segment of Λ. Let γ be an embedded path in $\widetilde{X}$ with the*

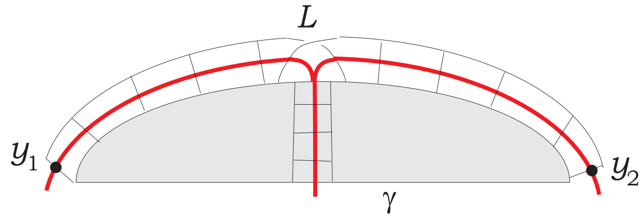

FIGURE 10.

same endpoints as λ. (Here γ is an edge path which starts and ends at "mid-edge vertices" corresponding to vertices of Λ.) Then there exists a reduced diagram F quasicollared by λ and γ.

Moreover, in the case γ does not intersect Λ anywhere except at its endpoints, then F is actually collared.

Remark 3.18. γ will be geodesic in our applications. This will serve to study the metric properties of embedded hypergraphs.

Proof. We proceed as above to get a reduced diagram quasicollared by λ and γ: Let L be the ladder carrying λ. Let $P \to L$ be a simple edge-path with the same endpoints as λ. Let $D \to \widetilde{X}$ be a disc diagram with boundary path $P^{-1}\gamma$. The 2-complex $F = D \cup_P L$ is a quasicollared diagram between Λ and γ.

The reduction process is carried out as above. Note that since λ embeds there is no pair of cancellable 2-cells between L and L, and so λ is preserved by the reduction process.

Now suppose that F is not a *collared* diagram. Then λ passes through an edge of P. Thus λ can be extended into D by a segment μ in D lying in the same hypergraph Λ (see Figure 10). Since Λ is an embedded tree, μ cannot cross λ again. So it has to exit D by crossing γ, thus providing a third intersection point between Λ and γ. □

4. THE HYPERGRAPHS ARE EMBEDDED TREES AT $d < 1/5$

Henceforth, $\widetilde{X}$ is the Cayley 2-complex associated to a finite presentation of the random group G at density d and length ℓ (Def. 1.1), that is, $\widetilde{X}$ is the universal cover of the standard 2-complex associated to the presentation. In case ℓ is odd, we perform a subdivision of all 1-cells of $\widetilde{X}$ so that hypergraphs can be defined.

Definition 4.1. Let D be a van Kampen diagram. The *external 1-cells* of D are the 1-cells which lie in ∂D. The other 1-cells are *internal*. A 2-cell of D is *external* if its closure contains an external 1-cell, otherwise it is *internal*.

A *pseudoshell* of D is a 2-cell R such that $|\partial R \cap \partial D| > \frac{1}{2}|\partial R|$.

A *shell* of D is a 2-cell R such that the boundary path of D contains a subpath Q, where Q is a subpath of the boundary path of R, and $|Q| > \frac{1}{2}|\partial R|$.

A *spur* of D is an 1-cell ending at a valence 1 0-cell on ∂D. Note that D has no spur if and only if its boundary path is immersed.

The following frequently arising condition is a special case of Greendlinger's lemma for $C'(\frac{1}{6})$ presentations:

Condition 4.2. For every reduced spurless van Kampen diagram $D \to X$ either

(1) D has at most one 2-cell.
(2) D contains at least two shells.

Theorem 4.3. *Let X be the standard 2-complex of some presentation. Suppose that X satisfies Condition 4.2. Then there is no reduced collared van Kampen diagram $D \to X$ (either cornerless or with a corner)*

Consequently, all hypergraphs are trees embedded in $\widetilde{X}$.

Proof. We show that there is no collared diagram. Indeed, suppose there is a collared diagram. It has no spurs, and has more than one 2-cell. But its only possible shell is its corner. Indeed, every other external 2-shell R, contains an edge of a hypergraph in the interior of D, so any path on $\partial R \cap \partial D$ has length $< \frac{1}{2}|\partial R|$. This contradicts Condition 4.2. □

Corollary 4.4. *For random groups at density $d < 1/5$, with overwhelming probability all hypergraphs are trees embedded in $\widetilde{X}$.*

Proof. Theorem 6 in [Oll07] states that Condition 4.2 holds with overwhelming probability for random groups at density $d < 1/5$. We can therefore apply Theorem 4.3. □

The goal of section 11 is to prove that as soon as $d > 1/5$, on the contrary there is only one hypergraph, which crosses every 1-cell of $\widetilde{X}$. Figure 21 at the end of the paper shows why hypergraphs are not embedded trees at $d > 1/5$.

We now turn to the metric aspect of the embeddings.

Theorem 4.5. *Consider a random group at density $d < 1/5$. With overwhelming probability, the distance in $\widetilde{X}^1$ between two vertices of a hypergraph Λ is at least $(1/2 - 2d - \varepsilon)\ell$ times the minimal number of edges joining them in Λ.*

Proof. Let γ be a geodesic in $\widetilde{X}^1$ between two points $y_1, y_2 \in \Lambda$. It is sufficient to prove the statement of the theorem under the additional hypothesis that γ does not intersect Λ at any other points.

By Lemma 3.17, there exists a reduced diagram E collared by γ and a ladder L carrying a segment of Λ.

Since E is collared and not only quasicollared, in particular it is an ordinary van Kampen diagram. Let n be the number of cells in the ladder L. We have $|\partial E| \leqslant n\ell/2 + |\gamma|$. But by Theorem 1.6, up to some ε we have

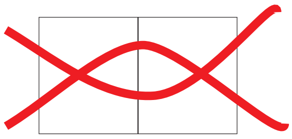

FIGURE 11.

$|\partial E| \geqslant (1-2d)\ell\,|E| \geqslant (1-2d)n\ell$ and so as claimed we have:

$$|\gamma| \geqslant n\ell(\frac{1}{2} - 2d)$$

□

Note that the multiplicative constant does not vanish as $d \to \frac{1}{5}$ (compare Figure 18 at $d > \frac{1}{5}$). However in the proof of this theorem, we already used that hypergraphs are embedded trees (a condition needed in our previous study of diagrams collared between a hypergraph and a geodesic).

Corollary 4.6. *In random groups at density $d < \frac{1}{5}$, with overwhelming probability, the stabilizer of any hypergraph is a free, quasiconvex subgroup.*

Proof. Since hypergraphs are trees in the Cayley complex, their stabilizers acts freely on a tree. Since random groups are torsion-free, so are the stabilizers, hence freeness since torsion-free groups acting freely on trees are free. Now a quasi-isometrically embedded tree in a hyperbolic space is quasiconvex, since quasi-geodesics remain at bounded distance from geodesics. □

5. 2-COLLARED DIAGRAMS

In Section 4, we showed that hypergraphs do not self-intersect at density $d < \frac{1}{5}$. It will also be useful to understand the way a pair of hypergraphs can intersect each other. A naive hope would be that distinct hypergraphs are either disjoint or intersect in a single point, but this is almost never the case as Figure 11 shows. However, we will show that at low density, intersecting hypergraphs might "braid" with each other a bit, but after departing do not converge again, so that intersection is a relatively local matter.

In Theorem 4.3 we saw that there are no 1-quasicollared diagrams at $d < \frac{1}{5}$. We now turn to 2-collared diagrams.

Theorem 5.1. *For random groups at $d < \frac{1}{6}$, with overwhelming probability every reduced diagram with at least three 2-cells has at least three pseudoshells.*

In particular, there exists no reduced 2-collared diagram except the one depicted in Figure 11.

Corollary 5.2. *For random groups at density $d < \frac{1}{6}$, with overwhelming probability the following holds: Let Λ_1, Λ_2 be two hypergraph rays intersecting in 2-cell c. Either they intersect in a 2-cell adjacent to c as in Figure 11, or they do not intersect anywhere else.*

Proof of the corollary. This follows from the theorem by Theorem 3.12. □

To prove the theorem, we shall need the following lemma (which will be of independent use).

Lemma 5.3. *Consider a random group at density $d < 1/4$. With overwhelming probability the following holds.*

Let D be a reduced spurless van Kampen diagram. Let n_{ps} be the number of pseudoshells in D. Let n_{i} be the number of internal 2-cells of D.

Then the number of external 2-cells in D is at most

$$\frac{(1/2-d)(n_{\mathrm{ps}}/2-n_{\mathrm{i}})}{1/4-d}$$

Proof of the lemma. Let A be the set of 2-cells R of D with $\partial R \cap \partial D \neq \varnothing$. Let $B \subset A$ be those 2-cells R with $|\partial R \cap \partial D| > \ell/2$ (the pseudoshells of D). Let $C \subset B$ be those 2-cells R with $|\partial R \cap \partial D| > \ell(1-d)$. Let $n_{\mathrm{e}} = \#A$, $n_{\mathrm{ps}} = \#B$ and $n_d = \#C$.

Let D' be the diagram obtained from D by removing the 2-cells in C. (D' might not be connected, but this does not matter since Theorem 1.6 applies to non-connected diagrams as well.) Let us evaluate the boundary length of D'. By definition, a 2-cell in C contributes at most $d\ell$ edges to $\partial D'$ (the ones that were not on ∂D). All other edges of $\partial D'$ were already present on ∂D and belonged to the boundary of a 2-cell in $A - C$. A 2-cell in $A - B$ contributes at most $\ell/2$ edges and a 2-cell in $B - C$ contributes at most $\ell(1-d)$ edges. So we have

$$\left|\partial D'\right| \leqslant q d\ell + (n_{\mathrm{e}} - n_{\mathrm{ps}})\frac{\ell}{2} + (n_{\mathrm{ps}} - n_d)(1-d)\ell.$$

On the other hand, by Theorem 1.6,

$$\left|\partial D'\right| \geqslant (1-2d-\varepsilon)\ell\left|D'\right| = (1-2d-\varepsilon)(n_{\mathrm{e}} + n_{\mathrm{i}} - n_d)\ell$$

and the combination of these two inequalities yields the conclusion. □

Proof of Theorem 5.1. Suppose there are at most two pseudoshells. By Lemma 5.3, the number of external 2-cells is bounded above by $\frac{(1/2-d)(n_{\mathrm{ps}}/2-n_{\mathrm{i}})}{1/4-d}$. When $n_{\mathrm{ps}} = 2$ and $d < \frac{1}{6}$ this bound is < 4 and so there are at most 3 external 2-cells. Note that if $n_{\mathrm{i}} \geqslant 1$ then there are $\leqslant 0$ external 2-cells, hence $n_{\mathrm{i}} = 0$ and there are no internal 2-cells.

So it is enough to rule out diagrams D having exactly three 2-cells r_1, r_2, r_3 where only r_1, r_2 are pseudoshells. Since r_3 is not a pseudoshell the internal length of D is at least $\ell/2$ and so $|\partial D| < |D|\,\ell - 2(\ell/2) = 2\ell$. But for $d < 1/6$ Theorem 1.6 yields $|\partial D| > \frac{2}{3}\,\ell\,|D| = 2\ell$ (choosing e.g. $\varepsilon < (1/6-d)/10$) hence a contradiction.

Note that a 2-collared diagram has at most two pseudoshells (its corners). □

Another consequence of the lemma is the following.

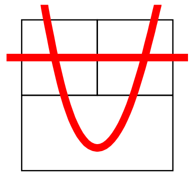

FIGURE 12.

Theorem 5.4. *For random groups at $d < \frac{1}{5}$, with overwhelming probability, any reduced 2-collared diagram has at most five 2-cells, and no internal cells.*

Proof. For $d < \frac{1}{5}$ and $n_{\mathrm{ps}} = 2$, the quantity $\frac{(1/2-d)(n_{\mathrm{ps}}/2-n_{\mathrm{i}})}{1/4-d}$ is less than 6, and non-positive if $n_{\mathrm{i}} \geqslant 1$. □

A less sharp version of this last assertion probably follows from the quasiconvexity obtained in Section 4.

6. TYPICAL CARRIER 2-CELLS AT $d < 1/5$

We saw in Theorem 5.1, that at densities less than 1/6, there are no nondegenerate 2-collared diagrams, whereas it is not difficult to check that as soon as $d > 1/6$ there are 2-collared diagrams with more than two 2-cells (e.g. the one of Figure 12). However, as proven in this section, for "most" 2-cells of $\widetilde{X}$, there are no 2-collared diagrams having these 2-cells as corner cells.

Let $(r_1, \ldots, r_N)$ be the N-tuple of random relators making the presentation, where by definition $N = (2m-1)^{d\ell}$. In the sequel we prove that some bad properties are excluded for relator r_1 with overwhelming probability (these properties are excluded with high probability for any relator r_i with i *fixed in advance*; however, for any random sample $(r_1, \ldots, r_N)$, there might be some i depending on the random sample, such that r_i satisfies these bad properties).

Lemma 6.1. *Consider a random group at density d. Then with overwhelming probability the following holds.*

Let D be a reduced diagram with $|D| = 3$, and suppose D contains a 2-cell corresponding to relator r_1. Then the number of internal 1-cells in D is at most $2d\ell + \varepsilon\ell$.

In particular at $d < 1/4$ the diagram of Figure 12 does not contain the relator r_1.

Proof. By Theorem 1.12, the expected number of fulfillings of D is at most

$$\mathbb{E}S_n(D) \leqslant (2m-1)^{\frac{1}{2}(|\partial D|-(1-2d)\ell|D|)} = (2m-1)^{d\ell|D|-L}$$

where $L = \frac{1}{2}(\ell\,|D| - |\partial D|)$ is the internal length of D.

By symmetry of all relators in the presentation, the expected number of fulfillings of D having the fixed relator r_1 as one of its 2-cells is at most

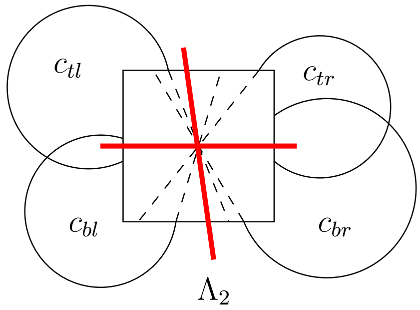

FIGURE 13.

$3(2m-1)^{-d\ell}\mathbb{E}S_n(D)$ (the 3 accounts for the choice of the 2-cell in D we are talking about). Thus the probability that there exists such a fulfilling is at most

$$3(2m-1)^{-d\ell}\mathbb{E}S_n(D) \leqslant 3(2m-1)^{2d\ell-L}$$

which decreases exponentially fast if $L \geqslant 2d\ell + \varepsilon\ell$. □

Lemma 6.2. *Consider a random group at $d < 1/4$. Let c be a 2-cell in $\widetilde{X}$ mapping to r_1. Let Λ_1 be any hypergraph through c. Then, with overwhelming probability, there exists a hypergraph Λ_2 through c which is locally transverse to Λ_1 in the following sense:*

If Λ_1 (resp. Λ_2) intersects ∂c at the points x_l, x_r (resp. x_t, x_b), then any 2-cell adjacent to c contains at most one of x_l, x_r, x_t, x_b. In particular, Λ_1, Λ_2 do not form a 2-collared diagram as illustrated in Figure 11.

Moreover, there are at least $(1/2 - 2d - \varepsilon)\ell$ choices for Λ_2.

Proof. There are two paths p_{top}, p_{bot} from x_l to x_r in ∂c. Note that since a random word of large length ℓ very probably contains every generator of the group at least once, every 1-cell in $\widetilde{X}$ is contained in several 2-cells). Now define c_{tl} ("topleft") as any 2-cell adjacent to c and containing x_l so that the length of the intersection of ∂c_{tl} with p_{top} is maximal. (Note that thanks to Corollary 1.11, the intersection of ∂c_{tl} with ∂c is connected.) Let ℓ_{tl} be this length.

Define c_{tr}, c_{bl}, c_{br} and $\ell_{tr}, \ell_{bl}, \ell_{br}$ similarly (see Figure 13). We may have $c_{tl} = c_{bl}$ and $c_{tr} = c_{br}$, but this does not affect our argument.

We have $\ell_{tl} < 2d\ell - \ell_{bl} + \varepsilon\ell$, otherwise the diagram $c \cup c_{tl} \cup c_{bl}$ would contradict Lemma 6.1. (In order to get a genuine van Kampen diagram in case $c_{tl} \cap c_{tb}$ contains some 1-cells, we have to unglue a bit c_{tl} below x_l and c_{bl} above x_l — this is consistent with our definition of ℓ_{tl} and ℓ_{bl} as the length of the intersection with resp. p_{top} and p_{bot}).

Similarly, we have $\ell_{tl} \leqslant 2d\ell - \ell_{tr} + \varepsilon\ell$, $\ell_{br} \leqslant 2d\ell - \ell_{bl} + \varepsilon\ell$, and $\ell_{br} \leqslant 2d\ell - \ell_{tr} + \varepsilon\ell$.

Set $L_1 = \max(\ell_{tl}, \ell_{br})$ and $L_2 = \max(\ell_{bl}, \ell_{tr})$. We have $L_1 \leqslant 2d\ell + \varepsilon\ell - L_2$. Since $d < 1/4$ we can choose ε so that $2d + \varepsilon < 1/2$, and so $L_1 < \ell/2 - L_2$ (and the discrepancy is at least $(1/2 - 2d - \varepsilon)\ell$).

Now take any point x_t on p_{top} so that the distance from x_t to x_l lies in the interval $\big(L_1, (\ell/2 - L_2)\big)$. There are at least $(1/2 - 2d - \varepsilon)\ell$ such points. Let $x_b \in p_{bot}$ be the opposite point in c. By construction, x_t and x_b do not lie in any of $c_{tl}, c_{tr}, c_{bl}, c_{br}$. By maximality of these latter 2-cells among 2-cells adjacent to c containing either x_l or x_p, no other 2-cell adjacent to c can contain two of the x's.

Now let of course Λ_2 be the hypergraph through x_t and x_b. □

Lemma 6.3. *Consider a random group at $d < 1/5$. With overwhelming probability, there is no reduced 2-collared diagram admitting r_1 on one of its corner cells, except the one on Figure 11.*

Proof. Let D be a 2-collared diagram having relator r_1 as one of its corner cells. By Theorem 5.4, we only have a finite number of diagrams to check. We can thus obtain overwhelming probability by intersecting finitely many events with overwhelming probability.

First, suppose that the other corner 2-cell of D has less than $(1-d)\ell$ edges on the boundary of D. Since in a 2-collared diagram, every 2-cell except maybe the corners has less than half its length on the boundary, this means that we have $|\partial D| \leqslant \ell + (1-d)\ell + (|D| - 2)\ell/2$. So the expected number of fulfillings of this diagram is, by Theorem 1.12, at most:

$$\mathbb{E}S_n(D) \leqslant (2m-1)^{\frac{1}{2}(|\partial D| - (1-2d)\ell|D|)} \leqslant (2m-1)^{\ell(1/2 - d/2 + |D|(d-1/4))}$$

By symmetry of all $(2m-1)^{d\ell}$ relators in the presentation, the expected number of fulfillings of D having the fixed relator r_1 as its corner 2-cell is at most $(2m-1)^{-d\ell}\mathbb{E}S_n(D)$, and so the probability that there exists such a fulfilling is at most

$$(2m-1)^{-d\ell}\mathbb{E}S_n(D) \leqslant (2m-1)^{\ell(1/2 - 3d/2 + |D|(d-1/4))}$$

so that if

$$1/2 - 3d/2 + |D|\,(d - 1/4) < 0$$

then this probability is exponentially small. So if $|D| > \frac{1/2 - 3d/2}{1/4 - d}$ then with overwhelming probability this does not happen. For $d < 1/5$ the right-hand side is less than 4. So the only possibility is the three 2-cell diagram depicted on Figure 12. But we have just excluded it in Lemma 6.1.

Second, suppose that the other corner of D has more than $(1-d)\ell$ on the boundary. Then we get the same conclusion by reasoning on the new diagram D' obtained by removing this corner. □

7. Codimension-1 subgroups at $d < 1/5$

Here for some time $\widetilde{X}$ is an arbitrary simply connected 2-complex each 2-cell of which has even boundary length; $\widetilde{X}$ is equipped with its hypergraph system as in Definition 2.1. We shall then return to the case when $\widetilde{X}$ is the Cayley 2-complex of a presentation of a random group at density d and length ℓ (up to subdivision of the 1-cells if ℓ is odd).

Definition 7.1. For a hypergraph Λ in the 2-complex $\widetilde{X}$, the *orientation-preserving stabilizer* $\text{Stabilizer}^+(\Lambda)$ of Λ is the index $\leqslant 2$ subgroup of $\text{Stabilizer}(\Lambda)$ that also stabilizes each of the two halfspaces which are components of $\widetilde{X} - \Lambda$. Equivalently, $\text{Stabilizer}^+(\Lambda)$ equals $\text{Stabilizer}(H^+)$ where H^+ is one of the components of $\widetilde{X} - \Lambda$.

We now prove the existence of codimension-1 subgroups at density $d < 1/5$. These subgroups are orientation-preserving stabilizers of hypergraphs passing through "typical" 2-cells of $\widetilde{X}$.

Lemma 7.2 (Codimension-1 criterion)**.** *Suppose that the discrete group G acts cocompactly on the simply connected 2-complex $\widetilde{X}$ and that the system of hypergraphs in $\widetilde{X}$ is locally finite and cocompact (meaning that the hypergraphs in $\widetilde{X}/G$ are compact and there is only a finite number of them). Suppose that two distinct hypergraphs Λ_1 and Λ_2 cross each other at a single point. Suppose that each Λ_i is an embedded tree with no leaves.*

Then $H_i = \text{Stabilizer}^+(\Lambda_i)$ is a subgroup of G with relative number of ends $e(G, H_i) = 2$.

Proof. Let $H_1 = \text{Stabilizer}^+(\Lambda_1)$. Let $\bar{X}_1 = H_1 \backslash \widetilde{X}$. We have to prove that $\bar{X}_1$ has at least two ends.

According to Lemma 2.3, Λ_1 separates $\widetilde{X}$ into two connected components. Let $\bar{\Lambda}_i$ be the image of Λ_i in $\bar{X}_1$. Suppose a component of $\bar{X}_1 - \bar{\Lambda}_1$ is compact. Consider the edge e of Λ_2 where Λ_2 crosses Λ_1. Using that Λ_2 is a leafless tree, extend e to a ray r in the direction of the halfspace mapping to the compact component of $\bar{X}_1 - \bar{\Lambda}_1$.

By compactness, the projection $\bar{r}$ of r to $\bar{X}_1$ must pass through $\bar{\Lambda}_1$ a second time. Indeed, let u be the first combinatorial subpath of the graph r whose initial and final vertices have the same projection in $\bar{X}_1$, so that the projection $\bar{u}$ is a cycle in $\bar{X}_1$; such a path necessarily exists by compactness. Thus $r = suv$, and s is minimal with this property. Consider the path $p = \bar{s}\bar{u}\bar{s}^{-1}$. We show that p is an immersed path. Indeed, if $\bar{u}\bar{s}^{-1}$ has a backtrack then $\bar{u} = \bar{w}\bar{e}$ and $\bar{s} = \bar{s}'\bar{e}$. Let $\bar{u}' = \bar{e}\bar{w}$. Then $\bar{u}'$ is a closed path with $|\bar{s}'| < |\bar{s}|$, so $\bar{u}'$ occurs earlier than $\bar{u}$ which contradicts the choice of u.

The lift $\tilde{p}$ of p to $\widetilde{X}$ is a segment of Λ_2, which is not closed since it is a subpath of r which is a geodesic in Λ_2.

Finally, let q_1 be a path in Λ_1 which projects to a path in $\bar{\Lambda}_1$ with the same endpoints as p. The common endpoints of p and q_1 provide, after lifting to $\widetilde{X}$, two intersections of the hypergraphs in $\widetilde{X}$, which contradicts the assumption. □

Lemma 7.3. *With overwhelming probability, at any density, the first relator r_1 in the random presentation involves all generators.*

Consequently, hypergraphs have no leaves, and any hypergraph passes through a 2-cell bearing relator r_1.

Proof. The first assertion is a consequence of the law of large numbers. It follows that any 1-cell of the Cayley 2-complex of a random group is contained in a 2-cell bearing relator r_1; hence, hypergraphs are leafless. □

Theorem 7.4. *With overwhelming probability, random groups* G *at density* $d < \frac{1}{5}$ *have a subgroup* H *which is free, quasiconvex and such that the relative number of ends* $e(G,H)$ *is at least* 2.

This subgroup can be taken to be the orientation-preserving stabilizer of any hypergraph.

Proof. Let r_1 be the first relator in the presentation. Let Λ_1 be a hypergraph. By Lemma 7.3 this hypergraph travels through a 2-cell c bearing r_1. Let Λ_2 be the hypergraph provided by Lemma 6.2. By Theorem 4.3, these hypergraphs are embedded trees, leafless by Lemma 7.3.

By Lemma 6.3 (in conjunction with Lemma 6.2), Λ_1 and Λ_2 do not form any reduced collared diagram with corner 2-cell c, and so by Theorem 3.12 they intersect only at c.

Now apply Lemma 7.2 to get the number of relative ends. The other assertions follow from Corollary 4.6. □

Corollary 7.5. *Suppose that* $d < 1/5$. *Then with overwhelming probability, a random group does not have Property* (T).

Proof. It was shown in [NR98] that groups having a subgroup with more than one relative end do not have Property (T). □

8. Carriers are convex at $d < 1/6$

Recall that the carrier of a hypergraph is the set of 2-cells the hypergraph passes through. We say that a subcomplex Y of the 2-complex $\widetilde{X}$ is convex if for any two 0-cells in Y, the shortest path between them in $\widetilde{X}$ is included in Y.

Theorem 8.1. *The following holds with overwhelming probability at* $d < \frac{1}{6}$: *For each hypergraph* Λ *its carrier* Y *is a convex subcomplex of* $\widetilde{X}$.

Proof. Let y_1, y_2 be two points on Y (which may not lie on the same side of Λ). Let γ be a geodesic in $\widetilde{X}$ joining y_1 to y_2. We want to show that γ lies in Y.

Suppose that γ does not lie in Y. We can decompose γ into subparts which either are included in Y, or intersect Y only at their endpoints. There is nothing to prove in the former case, so we can suppose that the intersection of γ with Y is exactly $\{y_1, y_2\}$ (and in particular y_1 and y_2 lie on the same side of Λ).

Let L be a ladder in Y between 2-cells containing y_1 and y_2, and let y_1' and y_2' be the extremal points of the hypergraph segment contained in L. Let

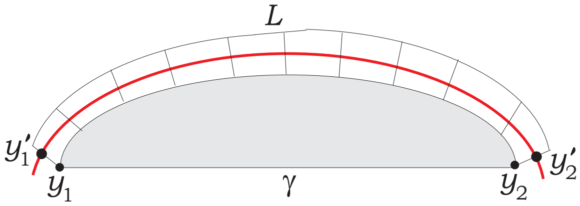

FIGURE 14.

$\gamma' = [y_1'y_1].\gamma.[y_2y_2']$ be the union of γ with paths joining y_i to y_i' respectively. Let D be a reduced van Kampen diagram collared by L and γ', as provided by Lemma 3.17 (see Figure 14). According to this lemma, D is collared and not only quasicollared since γ' does not intersect the hypergraph except at its endpoints.

Thanks to the collaring, every 2-cell of L except the two extremal ones has less than half its length on the boundary of D.

Now let c be a 2-cell lying on the boundary of D but not belonging to L (so that $\partial c \cap \partial D \subset \gamma$). Since γ is a geodesic, this means that the length of $\partial c \cap \gamma$ is no more than half the boundary length of c (otherwise we could shorten the geodesic).

So every 2-cell of D except maybe the two extremal cells of L has no more than half its length on the boundary of D, so that D has at most two pseudoshells. But at $d < 1/6$ this is ruled out by Theorem 5.1, except when $D = L$ has only two 2-cells, which was to be proven. □

Remark 8.2. At density $d < 1/6$, if a 2-cell R is such that ∂R is included in the carrier Y of some hypergraph Λ, then R itself is included in Y. Indeed, otherwise any hypergraph through R would meet Λ twice, contradicting Corollary 5.2.

Remark 8.3. It is not difficult to see that carriers are not convex when $d > 1/6$. Indeed, at density $> d$, with overwhelming probability there exists a diagram as depicted on Figure 15, in which the bottom 2-cell has more than half its boundary length on the boundary of the carrier, thus making it shorter to turn around from below. To see that such a diagram exists, first select the two middle relators of the diagram: they have to share a length $2d\ell$, and at density $> d$ such a pair of relators exists with overwhelming probability by Proposition 10 in [Oll05b]; then one has to find the right and left relators, each of which has to share a length $d\ell$ with the rest of the diagram, and at density $> d$ this is possible thanks to Proposition 9 in [Oll05b].

Remark 8.4. At density $d < \frac{1}{4}$, any 2-cell R is convex. Indeed, let p be a immersed path in ∂R with $|p| \leqslant \ell/2$, and let p' be a distinct immersed path with the same endpoints as p and $|p'| \leqslant |p|$. Then $p'p^{-1}$ is the boundary of a van Kampen diagram D, which thus satisfies $|\partial D| \leqslant \ell$. But according to

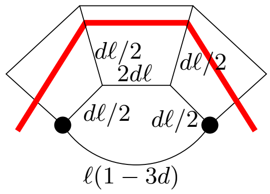

Figure 15. The carrier is not convex at $d > 1/6$.

Theorem 1.6 at $d < 1/4$, this implies that D has at most one 2-cell, which easily implies that $D = R$ and $|p| = \ell/2$.

9. Separation by hypergraphs at $d < 1/6$

The goal of this section is to prove that for any two points in $\widetilde{X}$, the number of hypergraphs separating them grows linearly with their distance. This will enable us to apply a CAT(0) criterion in Section 10.

For $p, q \in X$ we let $\#(p, q)$ equal the number of hypergraphs Λ such that p and q lie in distinct components of $X - \Lambda$.

Theorem 9.1. *The following holds with overwhelming probability at $d < \frac{1}{6}$: For all $p, q \in \widetilde{X}^0$ we have:*

$$\#(p, q) \geqslant \frac{1}{2}\left(\frac{1}{6} - d - \varepsilon\right)(d(p, q) - 6\ell)$$

Corollary 9.2. *A random group at density $d < 1/6$ has the Haagerup property.*

Proof of the corollary. A discrete group acts properly on its Cayley 2-complex (equipped, say, with the edge metric on the 1-skeleton and Euclidean metrics on each 2-cell). Now, since the hypergraphs are embedded trees, the system of hypergraph turns this 2-complex into a space with walls [HP98], and the theorem above states that the wall metric is equivalent to the edge metric. So the group acts properly on a space with walls, which by a folklore remark (see e.g. [CMV04]) implies the Haagerup property. □

For the proof of Theorem 9.1 we will need two lemmas.

Lemma 9.3. *The following holds with overwhelming probability at $d < \frac{1}{6}$: For each 2-cell R, any two disjoint pieces P_1, P_2 in ∂R satisfy $|P_1| + |P_2| < 3d\ell + \varepsilon\ell < \frac{\ell}{2}$.*

Proof. This follows directly from Theorem 1.6: indeed, we can form a three-2-cell diagram involving the two pieces, and at $d < 1/6$ its internal length is less than $\ell/2$. □

Lemma 9.4. *The following holds with overwhelming probability at $d < \frac{1}{6}$: Let Λ be a hypergraph passing through a 2-cell R. Then there exists another hypergraph Λ_d passing through R, such that $\Lambda \cap \Lambda_d$ consists of a single point.*

Actually there are at least $(1/2 - 3d - \varepsilon)\ell$ choices for Λ_d.

Sketch following Lemma 6.2. The proof is identical to Lemma 6.2, except that at density $< 1/6$ we do not have to fix the relator in advance (and we use Lemma 9.3 in place of Lemma 6.1). See Figure 13. □

Proof of Theorem 9.1. Let γ be a geodesic between p and q. We show that for each length-3ℓ subpath σ of γ, either the hypergraph through an edge at the middle of σ crosses γ only once, or there are at least $(1/2 - 3d - \varepsilon)\ell$ edges in σ the hypergraph through which crosses γ only once. This shows that whenever $d(p,q) \geqslant 6\ell$ the number of choices for such a hypergraph is at least $(1/2 - 3d - \varepsilon)\ell(6\ell)^{-1}\,(d(p,q) - 6\ell)$.

Let e_1 be an edge at the middle of σ. Let Λ_e be the corresponding hypergraph. If $\Lambda_e \cap \gamma = \Lambda_e \cap e_1$ then we are done. Otherwise Λ_e crosses γ at a first other edge e_3. Without loss of generality, assume that $e_1 < e_3$ in the ordering on γ.

By convexity, the subpath $[e_1 e_3]$ of γ lies in the carrier of Λ_e. Also remember that, by Remark 8.4, all 2-cells of $\widetilde{X}$ are convex.

First observe that e_1 and e_3 cannot be consecutive dual 1-cells of Λ_e crossing the same 2-cell R, for then $|[e_1 e_3]| = \frac{\ell}{2} + 1$ but the complementary part of ∂R has length $\frac{\ell}{2} - 1$ so γ would fail to be a geodesic.

In the other extreme, if there is more than one dual 1-cell of Λ_e between e_1 and e_3, then we let R be the second 2-cell in the ladder of Λ_e between e_1 and e_3. We then apply Lemma 9.4 to obtain a hypergraph Λ_k passing through R that intersects Λ_e at a single point (and we have $(1/2 - 3d - \varepsilon)\ell$ choices for Λ_k).

This hypergraph Λ_k crosses γ in a single edge k. Indeed, suppose Λ_k crossed γ in a second edge k_2. If $e_1 < k_2 < e_3$ then Λ_k crosses Λ_e in a second point which is impossible. Similarly, if $k_2 < e_1$ then by convexity $[k_2 k] \subset Y_k$ and hence $e_1 \subset Y_k$. Consequently, Λ_e crosses Λ_k at the center of a 2-cell containing e_1 on its boundary. By hypothesis, this 2-cell cannot be R, and so Λ_e and Λ_k intersect in more than one point which is impossible. An analogous argument excludes the possibility that $e_3 < k_2$.

Finally, we consider the case where e_1 and e_3 are separated by a single dual 1-cell e_2. Let R_e be the 2-cell between e_1 and e_2, and let R be the 2-cell between e_2 and e_3 (see Figure 16).

Applying Lemma 9.4, let Λ_f be a hypergraph passing through R that intersects Λ_e at a single point. Let Y_f be the carrier of Λ_f.

If Λ_f intersects γ in a single 1-cell f_3 then we are done (and there are $(1/2 - 3d - \varepsilon)\ell$ choices for Λ_f). Otherwise, let f_1 be the next 1-cell in γ that Λ_f passes through. As in the first case above, since γ is a geodesic, f_3 and f_1 cannot be consecutive dual 1-cells of Λ_f. We may also assume that there is no more than one dual 1-cell f_2 between them, for otherwise, as in the construction of Λ_k considered above, we could find a third hypergraph Λ_h intersecting Λ_f at a single point which is at the center of a 2-cell separating f_1 and f_3 and we would be done (once more with at least $(1/2 - 3d - \varepsilon)\ell$ choices for Λ_h).

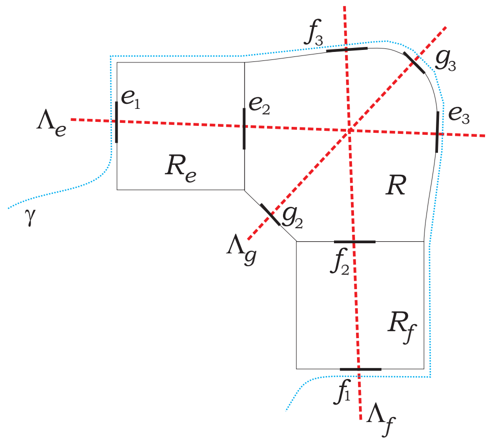

FIGURE 16.

Finally, we consider the case where there is exactly one dual 1-cell f_2 between f_3 and f_1 in Y_f. We refer the reader to Figure 16.

Let R_f be the 2-cell in Y_f between f_2 and f_1. Let $P_e = R \cap R_e$ and let $P_f = R \cap R_f$. Note that P_e and P_f are disjoint. Indeed, otherwise the complement of $\gamma \cap R$ is the concatenation of two pieces and this has length less than $3d\ell + \varepsilon\ell < \frac{\ell}{2}$ by Lemma 9.3, and so γ would not be a geodesic. Consequently, P_e and P_f are separated on either side of R by $R \cap \gamma$ and an edge g_2. More precisely, since $|\gamma \cap R| \leqslant \ell/2$ and $|P_e| + |P_f| < 3d\ell + \varepsilon\ell$, we have at least $(1/2 - 3d - \varepsilon)\ell$ choices for g_2.

Let Λ_g be the hypergraph dual to g_2, with carrier Y_g, and let g_3 be the 1-cell in ∂R that is antipodal to g_2. Note that since g_2 lies between e_2 and f_2, the 1-cell g_3 lies between f_3 and e_3 in ∂R, and hence Λ_g crosses γ in g_3. We will show that Λ_g does not cross any other edge of γ.

Note that g_2 is not a 1-cell of γ, since as γ is a geodesic it cannot contain two opposite 1-cells of a 2-cell.

By definition, g_2 does not lie in $R_e \cap R$. Neither does g_3, since f_3 is between e_2 and g_3 on ∂R and by definition Λ_f does not pass through R_e. So by Corollary 5.2, Λ_g does not pass through R_e, i.e. $R_e \not\subset Y_g$.

Let a be the last 1-cell in γ before R. Suppose that Λ_g crossed a 1-cell h of γ with $h \leqslant a$. Since R_e does not lie in Y_g we have $h \leqslant e_1$. But then since e_1 lies in the geodesic γ between g_3 and h, by convexity of Y_g we get $e_1 \subset Y_g$. Since $e_2 \subset Y_g$ too by construction, by convexity and Remark 8.2 we get $R_e \subset Y_g$ which is a contradiction.

Let b be the first 1-cell in γ after R. A similar argument shows that Λ_g cannot cross a 1-cell h of γ with $b \leqslant h$.

Note that we used only three 2-cells in the construction, so that the intersection of γ with these has length at most $3\ell/2$, hence all hypergraphs considered are dual to some edge inside the subpath σ; we have to consider a segment σ of length 3ℓ because we do not know on which side of e_1 the edge e_3 will fall. □

10. CAT(0) CUBULATION AT $d < 1/6$

We now proceed to the geometrization theorem at $d < 1/6$. We begin by listing the following criteria from [HW04]:

Theorem 10.1 (Local Finiteness). *Let $\widetilde{X}$ be a 2–complex equipped with a collection of hypergraphs satisfying the following properties.*

1. *$\widetilde{X}$ is locally finite.*
2. *The hypergraph system is uniformly locally finite.*
3. *There is a constant K so that for each $n \geq 1$, every pair of points at a distance at least nK apart are separated by at least n distinct hypergraphs.*
4. *There is a constant δ such that every hypergraph triangle is δ–thin.*

Then the cube complex C associated to $\widetilde{X}$ is locally finite.

Theorem 10.2 (Properness). *Let $\widetilde{X}$ be locally finite with a locally finite cube complex C. If Γ acts properly discontinuously on $\widetilde{X}$, then the induced action of Γ on C is also properly discontinuous.*

The following is formulated in [HW04] but was first proven by Sageev in [Sag97].

Theorem 10.3 (Cocompactness). *Suppose Γ acts properly and cocompactly on $\widetilde{X}$ then Γ acts properly and cocompactly on C provided that the following conditions hold.*

1. *$\widetilde{X}$ is δ–hyperbolic.*
2. *The hypergraphs are quasiconvex.*
3. *The hypergraph system is locally finite.*

We can now prove our second main theorem:

Theorem 10.4. *With overwhelming probability, a random group at density $d < \frac{1}{6}$ acts freely and cocompactly on a* CAT(0) *cube complex C.*

Proof. As shown by Gromov [Gro93] (see also [Oll04]), G is hyperbolic and torsion-free with overwhelming probability at $d < \frac{1}{2}$.

The quasiconvexity of the hypergraphs, and hence the codimension-1 subgroups that are their orientation-preserving stabilizers, was proven in Theorem 4.5 at $d < \frac{1}{5}$ with overwhelming probability. In fact, the convexity of hypergraph carriers was proven in Theorem 8.1 at $d < \frac{1}{6}$ with overwhelming probability.

The uniform local finiteness of $\widetilde{X}$ and the system of hypergraphs is obvious in our case.

Applying Theorem 10.3, we see that G acts cocompactly on the cube complex C.

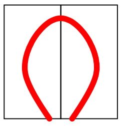

Figure 17. At $d > 1/4$, the hypergraph is trivially 1-dense in $\widetilde{X}$.

Since G is hyperbolic, and the hypergraphs embed by quasi-isometries, we see that all hypergraph triangles in $\widetilde{X}$ are δ-thin for some δ depending on the hyperbolicity constant for G and the quasi-isometry constants for the hypergraphs.

Finally the linear separation condition was proven in Theorem 9.1.

Thus the cube complex is locally finite by Theorem 10.1. Consequently G acts properly discontinuously on C by Theorem 10.2.

Since G is torsion-free we see that the action is free, and we are done. □

The crucial difference between the 1/5 and 1/6 cases was the separation of any two points by a linear number of hypergraphs proven in Theorem 9.1. We suspect that this should hold at density $d < 1/5$, but adapting the proof of Theorem 9.1 to this case involves the analysis of many particular cases corresponding to the existence of small 2-collared diagrams at density $1/6 < d < 1/5$.

Conjecture 10.5. *With overwhelming probability, random groups at density $d < \frac{1}{5}$ act freely and cocompactly on a CAT(0) cube complex.*

11. The unique hypergraph is π_1-surjective at $d > 1/5$

The next theorem shows that our approach fails at density $d > \frac{1}{5}$.

Theorem 11.1. *Let Λ be a hypergraph in the standard 2-complex $X = \widetilde{X}/G$ of a random group presentation at density $d > \frac{1}{5}$. Then $\pi_1\Lambda \to \pi_1 X \cong G$ is surjective.*

Proof. It is equivalent to prove that in $\widetilde{X}$, if the hypergraph Λ contains the midpoint of some edge e_1, then it also contains the midpoint of any other edge e_2 sharing a vertex with e_1. In particular it follows that there is only one hypergraph in $\widetilde{X}$.

First, let us give a very simple argument proving this under the stronger condition that $d > 1/4$. Indeed, in this case, with overwhelming probability there are two relators sharing a piece of length $\ell/2$. Thus the diagram of Figure 17 occurs with overwhelming probability (and then, since all reduced words have equal probability, every combination of generators of the group will occur on edges e_1 and e_2).

We will show that as soon as $d > 1/5$, with overwhelming probability there exists a van Kampen diagram as depicted on Figure 18, where edges e_1 and e_2 are the rightmost edges of the diagram. Actually, we will prove

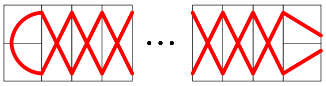

FIGURE 18. At $d > 1/5$, the hypergraph is 1-dense in $\widetilde{X}$.

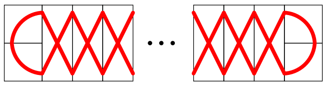

FIGURE 19.

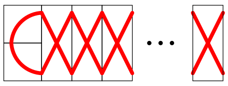

FIGURE 20.

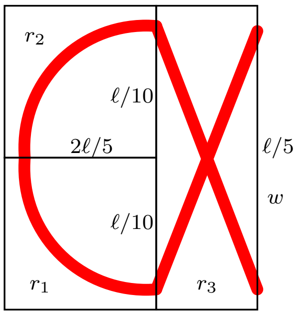

FIGURE 21. Hypergraphs do not embed at $d > 1/5$.

that at $d > 1/5$, very probably a van Kampen diagram as depicted on Figure 19 occurs; indeed, as the latter diagram has one more internal edge, it has smaller probability to be realized by random relators, hence if the latter diagram occurs with high probability then so does the former (and then, since all reduced words have equal probability, every combination of generators of the group will occur on edges e_1 and e_2).

The diagram in Figure 19 can be obtained by gluing two copies of the diagram in Figure 20. Let us divide the set of relators R of the random groups into two (arbitrary) halves R_1 and R_2 each with $\frac{1}{2}(2m-1)^{d\ell}$ relators. We will, at first, consider diagrams with relators only in R_1 or only in R_2.

Now fix an integer $K \geqslant 3$ and let D_K be the diagram as in Figure 20 with K 2-cells, and with the additional constraint that the "vertical" sides of the diagram have length $\ell/5$.

For $K = 3$ we get the diagram on Figure 21. In [Oll07] (last section), it is established that such a diagram occurs with overwhelming probability as soon as $d > 1/5$. This immediately proves that hypergraphs do not embed at $d > 1/5$, so that Corollary 4.4 is sharp. Note that $d = 1/5$ is the smallest value for which the topology of Figure 21 will yield a non-embedded hypergraph, in view of Theorem 1.6; note also that the results in [Oll07] are unrelated to hypergraphs but $d = 1/5$ already plays a special role.

We will, by induction, establish that for any $K < K_0 = 3 + \frac{1/5}{d-1/5}$ the diagram D_K occurs with overwhelming probability. More precisely, for a random group at $d > 1/5$ consider the set of all van Kampen diagrams with shape D_K occurring in the group and containing only relators from R_1, and let N_K be the number of distinct reduced words of length $\ell/5$ appearing on the rightmost side of all these diagrams. We are going to show that with overwhelming probability, $N_K \geqslant \frac{1}{8^{K-3}}(2m-1)^{(K-3)\ell(d-1/5)}$. We know that with overwhelming probability $N_3 \geqslant 1$.

We will use the following elementary lemma, the proof of which is omitted.

Lemma 11.2. *Let w be a random reduced word of length ℓ. Let w_1 and w_2 be disjoint subwords of w which are separated by at least one letter of w. Then the law of w_2 knowing w_1 is close to the law of a random reduced word independent from w_1.*

Here "close to" means that the ratio of probabilities lies between $1-c$ and $1+c$ for some small universal constant $c < 1$; moreover the error decreases exponentially with the number of letters separating w_1 and w_2.

Now, for $K \geqslant 4$ the diagram D_K is obtained from D_{K-1} by gluing a relator along a subword of length $\ell/5$ on the rightmost side. If the subword on the rightmost side of D_{K-1} is fixed, a random relator has probability $(2m-1)^{-\ell/5}$ to fulfill this condition. Since there are by definition N_{K-1} possibilities for the subword on the rightmost side of D_{K-1}, a random relator has probability $N_{K-1}(2m-1)^{-\ell/5}$ to form the rightmost 2-cell of a diagram with shape D_K.

Since there are $\frac{1}{2}(2m-1)^{d\ell}$ random relators in R_1, on average $\frac{1}{2}N_{K-1}(2m-1)^{\ell(d-1/5)}$ distinct diagrams D_K occur. This is only an average, but standard (e.g. Chernoff-type) deviation results show that with overwhelming probability, at least $\frac{1}{4}N_{K-1}(2m-1)^{\ell(d-1/5)}$ distinct diagrams D_K occur. To compute N_K we must consider the set of all subwords of length $\ell/5$ on the rightmost side of all these diagrams. According to the lemma above we get $\frac{1}{4}N_{K-1}(2m-1)^{\ell(d-1/5)}$ independent reduced words with (almost) uniform distribution. All these random words are not necessarily distinct (there might be some repetitions), but for $K < K_0$ this number is negligible compared to the total number of reduced words so that repetitions are negligible and, with overwhelming probability, we get at least $\frac{1}{8}N_{K-1}(2m-1)^{\ell(d-1/5)}$ distinct reduced words. So $N_K \geqslant \frac{1}{8}N_{K-1}(2m-1)^{\ell(d-1/5)}$ as needed.

Choose $K = K_0 - 1$. We know that $N_K \geqslant \frac{1}{8^{K_0-4}}(2m-1)^{(K_0-4)\ell(d-1/5)} \gg (2m-1)^{\ell/10}$ (if $1/5 < d \leqslant 1/4$). So we can produce more than $(2m-1)^{\ell/10}$ van Kampen diagrams with shape D_K, with more than $(2m-1)^{\ell/10}$ distinct random reduced words of length $\ell/5$ on their rightmost side. For this we have been using only the relators in R_1; using the other half of the relators in the presentation, we get a second set of diagrams with the same characteristics, distinct (as van Kampen diagrams) from the first set. Since there are $(2m-1)^{\ell/5}$ reduced words of length $\ell/5$, by the probabilistic pigeon-hole principle (e.g. [Oll05b], p. 31) two random sets of more than $(2m-1)^{\ell/10}$ reduced words of length $\ell/5$ will have a non-trivial intersection with overwhleming probability. Thus, we can find two distinct copies of the diagram in Figure 20, sharing the same rightmost subword. These two diagrams can be joined to form the diagram in Figure 19. This ends the proof. □

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CNRS, UMPA, École normale supérieure de Lyon, 46, allée d'Italie, 69364 Lyon cedex 7, France
E-mail address: yann.ollivier@umpa.ens-lyon.fr

Dept. of Math., McGill University, Montreal, Québec, Canada H3A 2K6
E-mail address: wise@math.mcgill.ca